LEONHARDI EULERI
OPERA OMNIA

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TYPIS ET IN AEDIBUS B. G. TEUBNERI
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LEONHARDI EULERI

COMMENTATIONES ANALYTICAE

AD THEORIAM INTEGRALIUM ELLIPTICORUM PERTINENTES

EDIDIT

ADOLF KRAZER

VOLUMEN PRIUS

LIPSIAE ET BEROLINI
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VORWORT DES HERAUSGEBERS

In den 20. und 21. Band der I. Serie von *Leonhardi Euleri Opera omnia* sind solche Abhandlungen Eulers aufgenommen worden, welche sich mit Integralen beschäftigen, die wir elliptische nennen, weil das zur Rektifikation der Ellipse dienende zu ihnen gehört.

Dieses Integral zog dadurch, daß es sich nicht durch die bekannten Funktionen darstellen ließ, die Aufmerksamkeit der Mathematiker seit dem Ausgange des 17. Jahrhunderts in hohem Maße auf sich und so sehen wir auch Euler bald nach Beginn seiner mathematischen Laufbahn, 1733, mit ihm beschäftigt. In der Abhandlung 28 (des Eneströmschen Verzeichnisses), mit der der vorliegende Band beginnt, findet Euler, daß mit Hilfe der Rektifikation der Ellipse die Lösung einer gewissen Differentialgleichung erster Ordnung, bei der die Trennung der Variablen nicht möglich ist, konstruiert werden könne. Der nämliche Gedanke, die Rektifikation der Ellipse zur Lösung von Differentialgleichungen zu verwenden, beschäftigt ihn auch 1734 in der folgenden Abhandlung 52 und führt hier zur Lösung der Aufgabe, auf einer Schar von Ellipsen mit gleicher einen Achse und gemeinsamem Scheitelpunkte von diesem aus gleichlange Bogenstücke abzuschneiden. Nach dieser Abhandlung tritt eine lange Pause ein und wir sehen erst 1749 Euler wieder mit dem Rektifikationsproblem der Ellipse beschäftigt; in 154 gibt er eine Reihenentwickelung für den Umfang der Ellipse.

Die geringe Zahl dieser Abhandlungen und auch die Art ihrer Problemstellung läßt erkennen, daß Euler zu einer fruchtbaren Entwickelung seiner Untersuchungen über elliptische Integrale einer Anregung von außen bedurfte, und wir wissen auch, wann und wie sie ihm geworden ist. Am 23. Dezember 1751 wurde er von der Berliner Akademie beauftragt, die ihr von Fagnano übersandten *Produzioni* zu prüfen, ehe man dem Verfasser antworte, und schon am 27. Januar 1752 liest Euler in der Akademie eine Abhandlung, 252, in welcher er für die auf die Ellipse und Hyperbel bezüglichen Resultate Fagnanos eine einfachere Ableitung gibt, die auf die Lemniskate bezüglichen aber wesentlich erweitert. Euler erfaßt sogleich die Bedeutung dieser Untersuchungen für die Integralrechnung. Am

Schlusse bemerkt er, daß auf diese Weise für zahlreiche Differentialgleichungen partikuläre Integrale gewonnen und damit der Weg zur vollständigen Lösung geebnet werde, und er gibt in der Tat 1753 in der jetzt folgenden Abhandlung 251 (daß diese später als 252 verfaßt ist, folgt auch aus den Angaben ENESTRÖMS) das allgemeine Integral der lemniskatischen Differentialgleichung an. Zu Beginn von 263 spricht sodann EULER das Problem, das für die Integralrechnung vorliegt, in seiner Allgemeinheit klar aus, nämlich zu zwei gegebenen Integralen $\int X dx$ und $\int Y dy$, die sich einzeln nicht integrieren lassen, eine solche algebraische Beziehung zwischen x und y zu finden, durch welche die beiden Integrale einander gleich werden oder sich um eine angebbare Größe unterscheiden. EULER erkennt aber zugleich, daß der umgekehrte Weg leichter zum Ziele führe, nämlich von einer algebraischen Gleichung zwischen x und y auszugehen und nach den Integralen zu fragen, die durch diese Gleichung miteinander verglichen werden können. Abhandlung 263 behandelt dann solche Relationen zwischen x und y, die zur Vergleichung von Kreis- und Parabelbogen, 261 solche, die zur Vergleichung von Ellipsen- und Hyperbelbogen dienen. Durch 261 ist zugleich die Lösung jenes Problems und der Beweis jenes Satzes vorbereitet, die EULER im Jahre vorher in 211 den Mathematikern vorgelegt hatte, und beide werden nun in 264 ausführlich mitgeteilt. Eine Umarbeitung von 263, 261 und 264 ist die den *Opera postuma* entnommene Abhandlung 818; auch ist der Inhalt von 263 und 261 in den Kap. V und VI der Sectio secunda des 1. Bandes der *Institutiones calculi integralis* wiederholt. An die vorgenannten Abhandlungen EULERS schließen sich inhaltlich noch 347 von 1765, sowie 581 und 582 von 1775 an, von denen aber nur 581 einen bemerkenswerten Fortschritt enthält. Während nämlich EULER bisher ausschließlich Integrale $\int \frac{P dx}{\sqrt{X}}$ betrachtet, bei denen P eine ganze rationale Funktion von x ist, werden hier für P auch gebrochene Funktionen zugelassen; EULER fügt also nach unserer heutigen Ausdrucksweise zu den Additionstheoremen der elliptischen Integrale 1. und 2. Gattung hier das von Integralen 3. Gattung.

Noch ist die aus dem Jahre 1778 stammende Abhandlung 714 zu erwähnen, in der EULER zeigt, daß durch seine Methode auch partikuläre Integrale solcher Differentialgleichungen erhalten werden können, bei denen sich die Variablen nicht trennen lassen.

Wenn man die EULERSCHE Methode, von einer algebraischen Gleichung zwischen x und y auszugehen und von ihr aus indirekt zur Lösung von Differentialgleichungen zu gelangen, genauer betrachtet, so erkennt man bald, daß sie im Wesentlichen in der Auffindung und Benutzung eines integrierenden Faktors besteht. Dies hat auch EULER in dem Supplementum zum 3. Bande seiner *Institutiones calculi integralis* genau ausgeführt und er glaubte damit den letzten Grund für die Möglichkeit der Integration seiner Differentialgleichungen gefunden zu haben. Er war daher nicht wenig erstaunt, als er 1775 (EULER erwähnt die Untersuchungen von LAGRANGE erstmals in der 506 zeitlich vorangehenden Abhandlung

582, ohne aber auf ihren Inhalt, von dem er damals wohl noch gar nicht Kenntnis genommen hatte, einzugehen) erfuhr, daß LAGRANGE schon vor geraumer Zeit im 4. Bande der Miscellanea Taurinensia eine direkte Methode zur Integration seiner Differentialgleichungen mitgeteilt habe. Zwar hatte auch er ungefähr um dieselbe Zeit, 1765, in 345 eine solche angegeben, war aber wegen der Umständlichkeit der benutzten Substitutionen nicht davon befriedigt. Jetzt bemächtigte er sich in 506 und in einer weiteren, noch im selben Jahre 1777 verfaßten Abhandlung 676 der Methode von LAGRANGE und benutzte sie zur Integration der von ihm früher behandelten Diffentialgleichungen.

Eine zweite Gruppe von Arbeiten EULERS über elliptische Integrale wurde durch die um die Mitte des 18. Jahrhunderts erschienenen Abhandlungen von MACLAURIN und D'ALEMBERT veranlaßt, von denen der erstere mit geometrischen, der letztere mit analytischen Hilfsmitteln eine Anzahl von Integralen abgeleitet hatte, die sich durch einfache Substitutionen auf die Rektifikation der Ellipse und Hyperbel reduzieren lassen. An diese Abhandlungen knüpfte EULER an; die früheste durch sie hervorgerufene Abhandlung EULERS ist die aus dem Jahre 1759 stammende 295, auf die erst im nächsten Jahre 273 folgte. Beide beschäftigen sich mit den Integralen $\int\sqrt{\frac{f+gx^2}{h+kx^2}}\,dx$ und teilen diese in drei Klassen, je nachdem das Integral durch einen einzigen Kegelschnittbogen, durch einen solchen und eine algebraische Funktion, oder endlich durch zwei Kegelschnittbogen und eine algebraische Funktion ausgedrückt wird. Es liegen hier die Keime der Reduktion der elliptischen Integrale auf eine Normalform vor, sie kommen aber wegen des Überwucherns der geometrischen Vorstellungen nicht zur Entfaltung. EULER verschärfte diese Auffassung noch, indem er in der Folge nach Kurven suchte, deren Bogenelement durch eine passende Substitution in das einer Ellipse übergehe, und so die Übereinstimmung der Integrale ohne Hinzutritt einer algebraischen Funktion verlangte. Eingeleitet wurden diese neuen Untersuchungen durch die Abhandlung 590 des Jahres 1775. Diese gibt drei Theoreme an. Das erste, daß alle imaginären Größen, die „in calculo analytico" auftreten, stets in die Form $a+bi$ gebracht werden können, gehört nicht hierher; das zweite sagt aus, daß es außer dem Kreise selbst keine algebraische Kurve gebe, deren Bogen durch Kreisbogen allein, und das dritte, daß es keine solche Kurve gebe, deren Bogen durch einen Logarithmus allein dargestellt werden können. EULER fordert die Mathematiker auf, für diese Theoreme strenge Beweise zu liefern. Er zeigt dann 1776, wie man im Gegensatze dazu zu einer gegebenen Parabel (638), zu einer gegebenen Ellipse (639) und zu einer Kurve, deren Bogendifferential von der Form $\frac{v^{m-1}\,dv}{\sqrt{1-v^{2n}}}$ (645) ist, unendlich viele algebraische Kurven angeben könne, die das gleiche Bogendifferential besitzen, und untersucht in 633 die allgemeinen Bedingungen, unter denen die Bogendifferentiale zweier Kurven übereinstimmen. Für die in 638, 639, 633 behandelten Probleme gab EULER später, 1781,

in 781, 780, 782 neuerdings Lösungen und bei dieser Gelegenheit bemerkte er, daß das früher in 590 für den Kreis aufgestellte Theorem nicht richtig sei. In 783 werden vielmehr unendlich viele algebraische Kurven, die keine Kreise sind, angegeben, deren Bogendifferential dem eines gegebenen Kreises gleich ist. In der den *Opera postuma* entnommenen Abhandlung 817 wird das in Rede stehende Problem noch einmal und zwar für die Lemniskate, die Parabel und die Ellipse gelöst.

Vier Abhandlungen, die in den Bänden 20 und 21 Platz gefunden haben, sind noch nicht genannt. Alle vier haben das Gemeinsame, daß Reihenentwicklungen ihren hauptsächlichsten Inhalt bilden. Abhandlung 448 nimmt das schon in 154 behandelte Problem der Reihenentwicklung für den Ellipsenumfang wieder auf, 605 behandelt die Eigenschaften der elastischen Kurve, 624 gibt Reihenentwicklungen für die Oberfläche des schiefen Kegels und 819 solche für den Hyperbelbogen.

Unter den Manuskripten, die die Petersburger Akademie der Redaktion zur Verfügung gestellt hat, befinden sich die Originale der Abhandlungen 817 und 818, sowie der Summarien von 28, 251, 252, 261, 263 und 264. Diese Manuskripte stimmen mit den früheren Abdrucken überein; nur das Manuskript des Summariums von 28 ist bisher noch nicht abgedruckt gewesen und erscheint hier zum ersten Male (am Schlusse des Bandes 20, da dessen Anfang schon gedruckt war, als das Manuskript vorgefunden wurde).

Wenn man den Gesamtinhalt der EULERSCHEN Abhandlungen über elliptische Integrale und ihre Bedeutung für die spätere Entwicklung der Theorie derselben überblickt, so ist die in dem einen Teile dieser Arbeiten (insbesondere 252, 261, 581) niedergelegte Entdeckung der Additionstheoreme der Integrale EULERS gewaltiges und bleibendes Verdienst. Fragen wir aber, warum das in dem anderen Hauptteil der Abhandlungen (insbesondere 295, 273) behandelte Problem der Reduktion der Integrale auf feste Normalformen und der Zurückführung des allgemeinen Integrals auf diese, trotzdem es für EULER, der rechnen konnte und wollte wie kein anderer, wie geschaffen war, keine so glückliche Lösung gefunden hat, so müssen wir, wie schon oben erwähnt, dem Nichtloskommen von geometrischen Vorstellungen die Schuld geben. Ein entscheidender Fortschritt in dieser Richtung konnte erst geschehen, wenn die geometrische Grundlage, der allerdings die Lehre von den elliptischen Integralen bisher fast alles verdankte, zurücktrat und einer Behandlung der Integrale um ihrer selbst willen Platz machte; der dies leisten sollte, war schon an der Arbeit, als EULER 1783 die Augen schloß: LEGENDRE.

Karlsruhe, den 1. November 1912.

ADOLF KRAZER.

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Insunt in hoc volumine indicis ENESTROEMIANI commentationes
28, 52, 154, 211, 251, 252, 263, 261, 264, 273, 295, 345, 347, 448

SPECIMEN DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIALIUM SINE INDETERMINATARUM SEPARATIONE

Commentatio 28 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 6 (1732/3), 1738, p. 168—174

1. Indeterminatarum separationem in aequationibus differentialibus ideo tam sollicite desiderari, quod ex ea inventa aequationis constructio sponte fluat, cuique in his rebus exercitato satis perspectum esse arbitror. Integratio praeterea aequationum differentialium, siquidem succedit, optime indeterminatis separandis instituitur. Quanquam enim innumerabiles dantur aequationes, quarum integrales sine huiusmodi separatione inveniri possunt, cuiusmodi methodum exhibuit Celeb. IOH. BERNOULLI in Comm. nostrorum Tom. I pag. 167[1]), tamen eae aequationes omnes ita sunt comparatae, ut vel per se obvia sit indeterminatarum separatio, vel saltem ex ipsa integratione facile derivetur. Similis vero est etiam ratio constructionum, quibus adhuc usi sunt Analystae; sunt enim omnes huiusmodi, ut aequationis, si nullo alio modo indeterminatae a se invicem separari possunt, separatio tamen ex ipsa constructione proficiscatur. Hanc ob rem nullam adhuc exhiberi posse existimo aequationem differentialem construibilem, cuius separatio omnes vires eluderet.

2. Nuper[2]) autem in ellipsi rectificanda occupatus inopinato incidi in aequationem differentialem, quam ope rectificationis ellipsis construere poteram,

1) IOH. BERNOULLI, *De integrationibus aequationum differentialium, ubi traditur methodi alicuius specimen integrandi sine praevia separatione indeterminatarum*, Comment. acad. sc. Petrop. 1 (1726), 1728, p. 167; *Opera omnia* T. 3, p. 108. A. K.

2) L. EULERI Commentatio 11 (indicis ENESTROEMIANI): *Constructio aequationum quarundam differentialium, quae indeterminatarum separationem non admittunt*, Nova acta erud. 1733, p. 369; *LEONHARDI EULERI Opera omnia*, series I, vol. 22. A. K.

neque tamen indeterminatarum separatio nequidem ex ipso construendi modo inveniri poterit. Aequatio vero, quam obtinui, erat haec

$$dy + \frac{y^2 dx}{x} = \frac{x dx}{x^2 - 1},$$

RICCATIANAE fere similis et forte ad separandum aeque difficilis ac haec $dy + y^2 dx = x^2 dx$. Casus hic mihi primum vehementer paradoxus videbatur; at constructione attentius perspecta facile intellexi ex ea non solum separationem indeterminatarum non posse deduci, sed etiam, si alio modo separatio haec succederet, multo maiora secutura esse absurda, comparationem scilicet perimetrorum ellipsium dissimilium, quae, ut mihi quidem videtur, omnem analysin superat. Constructio autem ipsa perquam est facilis; perficitur enim elongatione infinitarum ellipsium alterutrum axem communem habentium et hanc ob rem consueto per quadraturas construendi modo longe est praeferenda.

3. Proponam igitur totam rem, prout ad eam perveni. Sit ACB (Fig. 1) quadrans ellipticus, cuius centrum C, semi-axes vero AC et BC. Ponantur $AC = a$ et $BC = b$ et ex A ducatur tangens indefinita AT ad eamque ex centro C secans quaecunque CT abscindens arcum $AM = s$ voceturque $AT = t$. Demisso ex M in AC perpendiculo vocetur $CP = x$; erit ex natura ellipsis $PM = \frac{b\sqrt{(a^2 - x^2)}}{a}$ atque ob analogiam $CP : PM = CA : AT$ habebitur

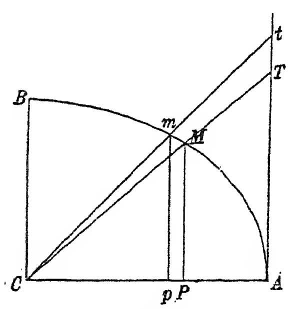

Fig. 1.

$$tx = b\sqrt{(a^2 - x^2)} \quad \text{seu} \quad x = \frac{ab}{\sqrt{(bb + tt)}}.$$

Sumatur arcus AM elementum Mm ducanturque mp, Ct prioribus MP, CT proximae; erit

$$Mm = ds = \frac{-dx\sqrt{(a^4 - (a^2 - b^2)x^2)}}{a\sqrt{(a^2 - x^2)}}$$

et $Tt = dt$. Quia autem est $x = \frac{ab}{\sqrt{(b^2 + t^2)}}$, erit $dx = \frac{-abtdt}{(b^2 + t^2)^{\frac{3}{2}}}$ et $\sqrt{(a^2 - x^2)} = \frac{at}{\sqrt{(b^2 + t^2)}}$ et $\sqrt{(a^4 - (a^2 - b^2)x^2)} = \frac{a\sqrt{(b^4 + a^2tt)}}{\sqrt{(b^2 + t^2)}}$. Ex his conficitur

$$ds = \frac{bdt\sqrt{(b^4 + a^2tt)}}{(bb + tt)^{\frac{3}{2}}}.$$

Ad cuius integrale per seriem saltem inveniendum pono $a^2 = (n+1)b^2$, quo prodeat

$$ds = \frac{b^2 dt \sqrt{((b^2+t^2)+nt^2)}}{(b^2+t^2)^{\frac{3}{2}}},$$

superiusque irrationale fit binomium, cuius alterum membrum est b^2+t^2 alterumque simplex terminus nt^2. Resolvo nunc $\sqrt{((b^2+t^2)+nt^2)}$ per canonem notum in seriem hanc

$$(b^2+t^2)^{\frac{1}{2}} + \frac{Ant^2}{(b^2+t^2)^{\frac{1}{2}}} + \frac{Bn^2t^4}{(b^2+t^2)^{\frac{3}{2}}} + \frac{Cn^3t^6}{(b^2+t^2)^{\frac{5}{2}}} + \text{etc.},$$

in qua brevitatis gratia est

$$A = \frac{1}{2}, \quad B = -\frac{1\cdot 1}{2\cdot 4}, \quad C = \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, \quad D = -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} \quad \text{etc.}$$

Habebitur ergo

$$ds = \frac{b^2dt}{b^2+t^2} + \frac{Ab^2nt^2dt}{(b^2+t^2)^2} + \frac{Bb^2n^2t^4dt}{(b^2+t^2)^3} + \frac{Cb^2n^3t^6dt}{(b^2+t^2)^4} + \text{etc.}$$

et integer arcus ellipticus s erit integrale huius seriei.

4. Notandum hic est singulorum horum terminorum integrationem ad primi termini $\int \frac{bbdt}{bb+tt}$ posse reduci; dat vero $\int \frac{bbdt}{bb+tt}$ arcum circuli radii b, cuius tangens est t. Hanc ob rem singulos terminos assumto hoc circulari arcu integrabo, ut sequitur:

$$\int \frac{b^2t^2dt}{(b^2+t^2)^2} = \frac{1}{2}\int \frac{bbdt}{bb+tt} - \frac{1}{2}\,\frac{b^2t}{bb+tt},$$

$$\int \frac{b^2t^4dt}{(b^2+t^2)^3} = \frac{1\cdot 3}{2\cdot 4}\int \frac{b^2dt}{bb+tt} - \frac{1\cdot 3}{2\cdot 4}\,\frac{b^2t}{bb+tt} - \frac{1}{4}\,\frac{b^2t^3}{(bb+tt)^2},$$

$$\int \frac{b^2t^6dt}{(b^2+t^2)^4} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int \frac{b^2dt}{bb+tt} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{b^2t}{bb+tt} - \frac{1\cdot 5}{4\cdot 6}\,\frac{b^2t^3}{(bb+tt)^2} - \frac{1}{6}\,\frac{b^2t^5}{(bb+tt)^3},$$

ex quibus lex integralium reliquorum terminorum iam satis apparet.

5. Si quarta perimetri ellipticae pars AMB requiratur, oportet facere t infinitum hocque facto omnes termini algebraici in superioribus integralibus

evanescunt. Arcus circularis vero $\int\frac{bb\,dt}{bb+tt}$ posito $t=\infty$ dabit quartam peripheriae circuli partem, cuius radius est b seu BC, quam designabimus littera e. Erit propterea

$$\int\frac{b^2dt}{bb+tt}=e,\quad \int\frac{b^2t^2dt}{(bb+tt)^2}=\frac{1\cdot e}{2},$$

$$\int\frac{b^2t^4dt}{(bb+tt)^3}=\frac{1\cdot 3\cdot e}{2\cdot 4},\quad \int\frac{b^2t^6dt}{(bb+tt)^4}=\frac{1\cdot 3\cdot 5\cdot e}{2\cdot 4\cdot 6}\ \text{etc.}$$

Prodibit igitur quarta perimetri ellipticae pars

$$AMB=e\left(1+\frac{1}{2}An+\frac{1\cdot 3}{2\cdot 4}Bn^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}Cn^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}Dn^4+\text{etc.}\right).$$

Atque substitutis loco A, B, C, D etc. valoribus debitis habebitur

$$AMB=e\left(1+\frac{1\cdot n}{2\cdot 2}-\frac{1\cdot 1\cdot 3\cdot n^2}{2\cdot 2\cdot 4\cdot 4}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot n^3}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot n^4}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}+\text{etc.}\right).$$

6. Haec series, si n est valde parvum seu $\frac{a^2-b^2}{b^2}$, id quod evenit, quoties ellipsis admodum propinqua est circulo, vehementer convergit; hocque casu igitur facile ellipsis perimeter invenitur. Quando vero n est quantitas quam minima seu $a=b+\omega$ denotante ω quantitatem quam minimam, erit $n=\frac{2\omega}{b}$ et $AMB=e\left(1+\frac{\omega}{2b}\right)$ quam proxime. Quando vero fit $a=0$, incidit punctum A in C et evadit $AMB=BC=b$; hoc vero casu erit $n=-1$; habebitur igitur

$$\frac{b}{e}=1-\frac{1}{2\cdot 2}-\frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\text{etc.}$$

Summa huius seriei ergo exprimit rationem radii ad quartam peripheriae partem in circulo.

7. Quemcunque igitur habeat valorem littera n in serie § 5 inventa, summa ... poterit assignari ope rectificationis ellipsis, cuius axis maior se ...m ut $\sqrt{(n+1)}$ ad 1. Hoc cum ita se habeat, usus sum ... mea summationes serierum ad resolutionem aequationum reducendi, quam nuper[1]) exhibui, ut investigarem, a cuius aequationis reso-

1) L. EULERI Commentatio 25 (indicis ENESTROEMIANI): *Methodus generalis summandi progressiones*, Comment. acad. sc. Petrop. 6 (1732/3), 1738, p. 68; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. A. K.

lutione summatio inventae seriei pendeat. Quo autem haec methodus facilius possit adhiberi, pono $n = -x^2$ eritque summanda ista series

$$1 - \frac{1 \cdot x^2}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot x^4}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^6}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \text{etc.};$$

huius igitur summam pono s. Erit ergo differentiando

$$\frac{ds}{dx} = -\frac{1 \cdot x}{2} - \frac{1 \cdot 1 \cdot 3 \cdot x^3}{2 \cdot 2 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} - \text{etc.}$$

Iam denuo per x multiplico sumoque differentialia posito dx constante; erit

$$\frac{d.\,xds}{dx^2} = -1 \cdot x - \frac{1 \cdot 1 \cdot 3 \cdot x^3}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4} - \text{etc.}$$

Porro divido ubique per x contraque per dx multiplico sumoque integralia; erit

$$\int \frac{d.\,xds}{xdx} = -x - \frac{1 \cdot 1 \cdot x^3}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4} - \text{etc.}$$

Denique iterum per dx multiplico, divido vero per x^3 et sumo integralia; erit

$$\int \frac{1}{x^3} \int \frac{d.\,xds}{x} = \frac{1}{x} - \frac{1 \cdot x}{2 \cdot 2} - \frac{1 \cdot 1 \cdot 3 \cdot x^3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{1 \cdot 1 \cdot 3 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \text{etc.}$$

Haec vero series est ipsa initialis per x divisa; eius igitur summa est $\frac{s}{x}$. Quocirca habemus hanc aequationem

$$\int \frac{1}{x^3} \int \frac{d.\,xds}{x} = \frac{s}{x},$$

quae sumtis differentialibus abit in hanc

$$x^2 ds - sx\,dx = \int \frac{d.\,xds}{x}.$$

Differentietur haec denuo; prodibit

$$x^2 dds + xdxds - sdx^2 = \frac{d.\,xds}{x} = dds + \frac{dxds}{x}.$$

Huius aequationis resolutio igitur pendet a summatione seriei propositae; quae cum per rectificationem ellipsis habeatur, aequationis constructio quoque dabitur.

8. Cum in ista aequatione s ubique unam teneat dimensionem, reduci ea poterit per methodum meam Tom. III Comm.[1]) insertam ad aequationem simpliciter differentialem facta substitutione $s = c^{\int p dx}$, ubi c denotat numerum, cuius log. est 1. Hoc posito erit $ds = c^{\int p dx} p dx$ et $dds = c^{\int p dx}(dp dx + pp dx^2)$ atque aequatio inventa transformabitur in hanc

$$x^2 dp + x^2 p^2 dx + px dx - dx = dp + pp dx + \frac{p dx}{x},$$

quae divisa per $xx - 1$ mutatur in istam

$$dp + pp dx + \frac{p dx}{x} = \frac{dx}{xx - 1}.$$

Ad hanc simpliciorem efficiendam pono $p = \frac{y}{x}$ et proveniet

$$dy + \frac{yy dx}{x} = \frac{x dx}{xx - 1}.$$

Quae quomodo separari possit, neque perspicio neque constructionis consideratio eo perducit.

9. Quo autem ipsa constructio huius aequationis ex praecedentibus deducatur, pono illum axis semissem AC, quem ante littera a denotavi, aequalem r, quia ut variabilis debet considerari, et quartam perimetri ellipsis partem respondentem q; erit $-xx = n = \frac{r^2 - b^2}{b^2}$ et $x = \frac{\sqrt{(b^2 - r^2)}}{b}$. Porro erit $q = es$; est vero $s = c^{\int p dx} = c^{\int \frac{y dx}{x}}$, quocirca habebitur $q = ec^{\int \frac{y dx}{x}}$ et $lq - le = \int \frac{y dx}{x}$ adeoque $y = \frac{x dq}{q dx} = \frac{(r^2 - b^2) dq}{q r dr}$. Ne autem, quando r maior est quam b, irrationalia proveniant, restituo loco xx valorem $-n$; erit $\frac{dx}{x} = \frac{dn}{2n}$ et $\frac{x dx}{xx - 1} = \frac{dn}{2(n+1)}$. His substitutis habebitur ista aequatio

$$2dy + \frac{y^2 dn}{n} = \frac{dn}{n+1},$$

1) L. Euleri Commentatio 10 (indicis Enestroemiani): *Nova methodus innumerabiles aequationes differentiales secundi gradus reducendi ad aequationes differentiales primi gradus*, Comment. acad. sc. Petrop. 3 (1728), 1732, p. 124; *Leonhardi Euleri Opera omnia*, series I, vol. 22. A. K.

quae construetur sumendis $n = \frac{r^2 - b^2}{b^2}$ et $y = \frac{(r^2 - b^2)\,dq}{qr\,dr}$ seu, iam invento n, $y = \frac{2n\,dq}{q\,dn}$. Hinc sequens nascitur constructio:

Descripto quadrante elliptico BCA (Fig. 2), cuius centrum in C et semiaxis BC constans est, puta $= 1$, pono hic 1 loco b, quo facilius homogeneitas possit servari. Erit ergo semi-axis $AC = r$; ex A erigatur normalis $AD =$ arcui elliptico AB; erit punctum D in curva aliqua BD, cuius constructio hoc modo est in promtu. In ea igitur erit $AD = q$. Sit F huius ellipsis focus; erit $CF = \sqrt{(r^2 - 1)}$; et ad BF ducatur normalis FP; erit $CP = r^2 - 1 = n$. Notetur hic, quando fit $AC < BC$ et focus F in BC incidit, valorem n fieri negativum et ex altera parte puncti C versus B accipi oportere. Deinceps ducatur tangens DT curvae BD in D; erit

$$AT = \frac{q\,dr}{dq};$$

et iuncta AP ex T ducatur recta TG normaliter secans AP, si opus est, productam in O et DA productae occurrens in G; erit ob similia triangula PCA et TAG

$$AG = \frac{rq\,dr}{(r^2 - 1)\,dq}.$$

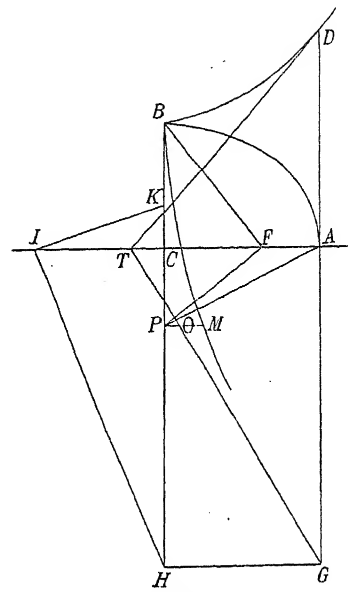

Fig. 2.

Ipsi AG aequalis capiatur CH et sumta $CI = CB = 1$ ad ductam HI erigatur perpendicularis IK; erit

$$CK = \frac{(r^2 - 1)\,dq}{rq\,dr} = y.$$

Huic CK fiat aequalis PM eritque M in curva quaesita BM; huius enim curvae haec est proprietas, ut dictis $CP = n$ et $PM = y$ sit

$$2\,dy + \frac{y^2\,dn}{n} = \frac{dn}{n+1}.$$

SOLUTIO PROBLEMATUM RECTIFICATIONEM ELLIPSIS REQUIRENTIUM

Commentatio 52 indicis ENESTROEMIANI
Commentarii academiae scientiarum Petropolitanae 8 (1736), 1741, p. 86—98

1. Agitata iam superiori seculo inter Geometras sunt huiusmodi problemata, in quibus linea curva requirebatur, quae ab infinitis positione datis curvis arcus aequales abscinderet. Communicaverunt etiam illo tempore Cl. Cl. Geometrae[1]) elegantes solutiones pro casu, quo curvae positione datae inter se sunt similes, uti cum ab infinitis circulis vel parabolis arcus aequales abscindendi essent. Nemo autem, quantum constat, ulterius est progressus neque quisquam pro curvis dissimilibus problemati satisfecit, etiamsi iam tum quaestio de infinitis ellipsibus proponeretur. Atque etiamnum, cum Insigni Geometrae per litteras significassem[2]) me aequationem pro curva, quae ab infinitis ellipsibus dissimilibus arcus aequales abscinderet, invenisse, ille mihi respondit huius problematis solutionem in sua non esse potestate meque simul rogavit, ut meam solutionem in non contemnendum analyseos augmentum communicarem.

2. Huius autem quaestionis summa difficultas in hoc consistit, quod diversarum et dissimilium ellipsium rectificationes a se mutuo non pendeant. Hanc enim ob causam curvae ab infinitis ellipsibus arcus aequales abscindentis

1) IAC. BERNOULLI, *Solutio sex problematum fraternorum in Ephem. Gall. 26. Aug. 1697 propositorum* (Probl. 4 et 5), Acta erud. 1698 p. 226; *Opera* p. 796; vide etiam IOH. BERNOULLI, *Opera omnia*, T. 1, p. 256. A. K.

2) L. EULER ad DAN. BERNOULLI, Novembri (?) 1734; vide G. ENESTROEM, *Der Briefwechsel zwischen LEONHARD EULER und DANIEL BERNOULLI*, Biblioth. Mathem. 7_3, 1906/7, p. 126, inprimis p. 140. A. K.

aequationem inventu maxime difficilem esse oportet, eo quod etiam concessa unius ellipsis rectificatione reliquarum tamen omnium rectificatio ab ista non pendeat. Deinde methodus, qua in huiusmodi problematis uti solent, ita est comparata, ut tantum ad curvas similes accommodari possit, pro curvis dissimilibus autem nullam afferat utilitatem.

3. Quod autem mihi primum viam ad huiusmodi difficilia problemata patefecit, est praecipue universalis mea series summandi methodus.[1]) Hac enim inventa statim[2]) aequationem differentialem, in qua indeterminatae nullo pacto a se invicem separari possunt, ope rectificationis ellipsium dissimilium construxi atque paulo post[3]) maxime agitatae aequationis RICCATIANAE constructionem et resolutionem communicavi.

4. Postmodum autem, cum haec per series operandi methodus nimis operosa et non satis genuina videretur, in aliam magis naturalem methodum et huius modi quaestionibus magis accommodatam inquisivi; atque tandem ex voto obtinui, ita ut eius beneficio non solum priora problemata, quae serierum ope resolveram, sed etiam innumera alia, ad quae tractanda series non sufficiunt, perficere potuerim. Methodum etiam hanc fuse exposui in dissertatione *De infinitis curvis eiusdem generis*[4]) anno praecedente [1734] proposita; quia vero, ne nimis essem prolixus, nulla adieci exempla, non satis apparet, quam late ea pateat quamque amplum in re analytica aperiat campum.

5. Quo igitur huius methodi vis et utilitas melius percipiatur, hac dissertatione eam ad infinitas ellipses accommodabo atque non solum monstrabo,

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1) Vide notam 1 p. 4. A. K.

2) L. EULERI Commentatio 28 (indicis ENESTROEMIANI); vide p. 1. A. K.

3) L. EULERI Commentatio 31 (indicis ENESTROEMIANI): *Constructio aequationis differentialis* $ax^n dx = dy + y^2 dx$, Comment. acad. sc. Petrop. 6 (1732/3), 1738, p. 231; *LEONHARDI EULERI Opera omnia*, series I, vol. 22. A. K.

4) L. EULERI Commentatio 44 (indicis ENESTROEMIANI): *De infinitis curvis eiusdem generis. Seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis*, Comment. acad. sc. Petrop. 7 (1734/5), 1740, p. 174; *LEONHARDI EULERI Opera omnia*, series I, vol. 22. A. K.

quomodo ab infinitis ellipsibus arcus aequales abscindi debeant, sed etiam innumerabilium tam primi quam secundi gradus aequationum differentialium resolutionem ope rectificationis ellipsium perficere docebo.

6. Quod enim ad curvam, quae ab infinitis ellipsibus arcus aequales abscindat, attinet, eius constructio eo ipso est facilis, quod ope rectificationis curvarum, quae facillime describi possunt, perfici queat. Atque hanc ipsam constructionem longe anteferendam esse censeo aliis per quadraturas curvarum peractis constructionibus. Non igitur tam illius curvae constructio requiritur quam eius aequatio, quo, quales aequationes tam facile construi queant, cognoscatur. Hanc ob rem analysis non parum augmenti accipiet, si illae aequationes proferantur, quae ope rectificationis ellipsium constructionem admittunt.

7. Considero igitur primum infinitas ellipses $AMDB$ (Fig. 1), quae omnes alterum axem, cuius semissis est CD, habeant eundem, axes vero transversos AB diversos. Si nunc vel ab his omnibus ellipsibus vel arcus aequales sint abscindendi vel in data ratione inaequales, vel curva sit invenienda, cuius constructio ope harum ellipsium quomodocunque praescribitur, ad talia problemata omnia solvenda opus est, ut aequatio habeatur inter arcum AM, abscissam AP et axem AB, in qua hae tres quantitates insint tanquam variabiles.

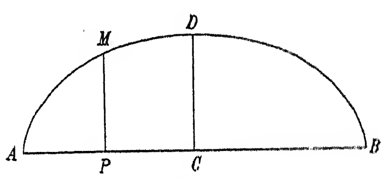

Fig. 1.

8. Huiusmodi ergo problematum solutio perficietur, si quaeratur aequatio modularis, quemadmodum in citata dissertatione de curvis infinitis eiusdem generis docui, inter arcum AM et abscissam AP et axem AB quoque variabilem. Quo igitur ad huiusmodi aequationem modularem perveniam, pono abscissam $AP = t$, applicatam $PM = u$, arcum $AM = z$, semiaxem variabilem $AC = a$, semiaxem constantem $CD = c$. His vero positis erit $u = \frac{c}{a}\sqrt{(2at - tt)}$ seu posito $t = ax$ erit $u = c\sqrt{(2x - xx)}$ atque $dt = adx$ et $du = \frac{cdx - cxdx}{\sqrt{(2x - xx)}}$. Ex his igitur fiet

$$dz = \frac{dx\sqrt{(2a^2x - a^2x^2 + c^2 - 2c^2x + c^2x^2)}}{\sqrt{(2x - xx)}}$$

positoque $a^2 - c^2 = b^2$ erit

$$z = \int \frac{dx\sqrt{[c^2 + b^2(2x - xx)]}}{\sqrt{(2x - xx)}}.$$

9. Integrali huic invento aequatur ergo z, si integratio fiat posito tantum x variabili, b vero et c constantibus. Praeterea in integratione talis addi debet constans, ut evanescat z posito $x = 0$. At quia aequatio desideratur, in qua a seu eius loco b aeque tanquam variabilis insit ac x et z, quaeritur aequatio differentialis, quae proditura esset, si

$$\int \frac{dx\sqrt{[c^2 + b^2(2x - xx)]}}{\sqrt{(2x - xx)}}$$

denuo differentietur posito praeter x etiam b variabili.

10. Ponatur nunc secundum methodum anno praeterito traditam x constans et differentietur quantitas $\frac{\sqrt{[cc + bb(2x - xx)]}}{\sqrt{(2x - xx)}}$; prodibit $\frac{bdb\sqrt{(2x - xx)}}{\sqrt{[cc + bb(2x - xx)]}}$. Quamobrem posito quoque b variabili erit

$$dz = \frac{dx\sqrt{[cc + bb(2x - xx)]}}{\sqrt{(2x - xx)}} + db\int \frac{bdx\sqrt{(2x - xx)}}{\sqrt{[cc + bb(2x - xx)]}},$$

quod postremum integrale ita debet accipi, ut evanescat posito $x = 0$; in eo vero iterum b tanquam constans inest. Ponatur brevitatis gratia

$$R = \frac{dz}{db} - \frac{dx\sqrt{[cc + bb(2x - xx)]}}{db\sqrt{(2x - xx)}};$$

erit

$$R = \int \frac{bdx\sqrt{(2x - xx)}}{\sqrt{[cc + bb(2x - xx)]}}.$$

11. Si nunc integrale, cui R aequatur, reduci posset ad integrationem formulae, cui z aequalis est, pro R inveniri posset valor finitus per z, qui substitutus in altera aequatione daret aequationem modularem quaesitam. Sed hae duae integrationes a se invicem non pendent, ut facile tentanti animadvertetur. Quamobrem ulterius progredi oportet et ultimam aequationem

denuo differentiare uti primam, ponendo quoque b variabilem. Invenietur autem hoc modo

$$dR = \frac{b\,dx\,\sqrt{(2x-xx)}}{\sqrt{[cc+bb(2x-xx)]}} + db\int\frac{cc\,dx\,\sqrt{(2x-xx)}}{[cc+bb(2x-xx)]^{\frac{3}{2}}},$$

quod integrale iterum ita accipi debet, ut evanescat posito $x=0$.

12. Ponatur iterum

$$S = \frac{dR}{db} - \frac{b\,dx\,\sqrt{(2x-xx)}}{db\,\sqrt{[cc+bb(2x-xx)]}};$$

erit

$$S = \int\frac{cc\,dx\,\sqrt{(2x-xx)}}{[cc+bb(2x-xx)]^{\frac{3}{2}}};$$

quae formula cum non sit integrabilis, videndum est, num eius integratio ab alterutra praecedentium vel ab utraque pendeat. Quod quo appareat, ponatur $S+\alpha R+\beta z = Q$, ubi α et β ab x et z sint quantitates liberae, Q vero utcunque ex x et b et constantibus composita; debet autem Q talis esse quantitas, ut evanescat posito $x=0$. Posito ergo b constante debebit esse $dQ = dS+\alpha dR+\beta dz$, ubi in differentiali ipsius Q b tanquam constans considerari debet.

13. At posito b constante est

$$dS = \frac{cc\,dx\,\sqrt{(2x-xx)}}{[cc+bb(2x-xx)]^{\frac{3}{2}}} \quad \text{et} \quad dR = \frac{b\,dx\,\sqrt{(2x-xx)}}{\sqrt{[cc+bb(2x-xx)]}}$$

et

$$dz = \frac{dx\,\sqrt{[cc+bb(2x-xx)]}}{\sqrt{(2x-xx)}}.$$

Hanc ob rem erit

$$\frac{dQ}{dx} = \begin{bmatrix} cc(2x-xx)+\alpha bcc(2x-xx)+\alpha b^3(2x-xx)^2 \\ +\beta c^4+2\beta b^2c^2(2x-xx)+\beta b^4(2x-xx)^2 \end{bmatrix} : [cc+bb(2x-xx)]^{\frac{3}{2}}\sqrt{(2x-xx)}.$$

Ponatur ad similem formam obtinendam $Q = \frac{(\gamma x+\delta)\sqrt{(2x-xx)}}{\sqrt{[cc+bb(2x-xx)]}}$, qui valor per se evanescit posito $x=0$.

14. Differentietur nunc Q posito tantum x variabili; erit

$$\frac{dQ}{dx} = [\gamma cc(2x - xx) + \gamma bb(2x - xx)^2 + \gamma ccx + \delta cc - \gamma ccx^2 - \delta ccx]$$
$$: [cc + bb(2x - xx)]^{\frac{3}{2}} \sqrt{(2x - xx)}.$$

Quia ergo denominatores iam sunt inter se aequales, fiant numeratores quoque aequales aequandis terminis, in quibus ipsius x similes sunt dimensiones; erit

$$\text{I. } \gamma bb = \alpha b^3 + \beta b^4$$
$$\text{II. } \gamma b^2 = \alpha b^3 + \beta b^4$$
$$\text{III. } 4\gamma bb - 2\gamma cc = 4\alpha b^3 + 4\beta b^4 - cc - \alpha bcc - 2\beta b^2 c^2$$
$$\text{IV. } 3\gamma cc - \delta cc = 2cc + 2\alpha bcc + 4\beta b^2 c^2$$

et

$$\text{V. } \delta cc = \beta c^4.$$

Hinc invenitur

$$\alpha = \frac{1}{b}, \quad \beta = \frac{-1}{b^2 + c^2}, \quad \gamma = \frac{cc}{bb + cc} \quad \text{et} \quad \delta = \frac{-cc}{bb + cc}.$$

15. His ergo valoribus substitutis prodibit

$$\frac{cc(x-1)\sqrt{(2x - xx)}}{(bb + cc)\sqrt{[cc + bb(2x - xx)]}} = S + \frac{R}{b} - \frac{z}{b^2 + c^2}.$$

Quia autem est

$$R = \frac{dz}{db} - \frac{dx\sqrt{[cc + bb(2x - xx)]}}{db\sqrt{(2x - xx)}} \quad \text{et} \quad S = \frac{dR}{db} - \frac{bdx\sqrt{(2x - xx)}}{db\sqrt{[cc + bb(2x - xx)]}}$$

atque

$$x = \frac{t}{a} \quad \text{et} \quad bb = a^2 - c^2 \quad \text{atque ideo} \quad bb + cc = a^2, \quad dx = \frac{ada - tda}{a^2} \quad \text{et} \quad db = \frac{ada}{b},$$

erit

$$Q = \frac{cc(t - a)\sqrt{(2at - tt)}}{a^3\sqrt{[a^2c^2 + (a^2 - c^2)(2at - tt)]}}$$

et

$$\frac{R}{b} = \frac{dz}{ada} - \frac{(adt - tda)\sqrt{[a^2c^2 + (a^2 - c^2)(2at - tt)]}}{a^3 da\sqrt{(2at - tt)}}$$

atque

$$S = \frac{c^2dz}{a^3da} + \frac{a^2-c^2}{a^2da}\,d.\frac{dz}{da} - \frac{a^2-c^2}{a^3da}\,d.\frac{dt}{da}\sqrt{\frac{a^2c^2+(a^2-cc)(2at-tt)}{2at-tt}}$$
$$+\frac{(2a^2-3c^2)(adt-tda)}{a^5da}\sqrt{\frac{a^2c^2+(a^2-cc)(2at-tt)}{2at-tt}}$$
$$-\frac{(2aa-2cc)(adt-tda)}{a^3da}\sqrt{\frac{2at-tt}{a^2c^2+(a^2-cc)(2at-tt)}}$$
$$+\frac{cc(a-t)(a^2-c^2)(adt-tda)^2}{a^3da^2(2at-tt)^{\frac{3}{2}}\sqrt{[a^2c^2+(a^2-cc)(2at-tt)]}}.$$

16. Ne autem in nimis prolixos calculos incidamus, retineamus literas b, x et z; erit

$$S = \frac{1}{db}\,d.\frac{dz}{db} - \frac{1}{db}\,d.\frac{dx}{db}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}} - \frac{2bdx}{db}\sqrt{\frac{2x-xx}{cc+bb(2x-xx)}}$$
$$+\frac{ccdx^2(1-x)}{db^2(2x-xx)^{\frac{3}{2}}\sqrt{[cc+bb(2x-xx)]}}.$$

His ergo loco S et R substitutis habebitur aequatio modularis ista

$$\frac{z}{bb+cc} = \frac{cc(1-x)\sqrt{(2x-xx)}}{(bb+cc)\sqrt{[cc+bb(2x-xx)]}} - \frac{dx}{bdb}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}}$$
$$-\frac{2bdx}{db}\sqrt{\frac{2x-xx}{cc+bb(2x-xx)}} + \frac{ccdx^2(1-x)}{db^2(2x-xx)^{\frac{3}{2}}\sqrt{[cc+bb(2x-xx)]}}$$
$$+\frac{dz}{bdb} + \frac{1}{db}\,d.\frac{dz}{db} - \frac{1}{db}\,d.\frac{dx}{db}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}}.$$

Atque haec est aequatio differentialis secundi gradus, in qua z, x et b aeque variabiles sunt positae. Ex hac autem aequatione sequentia problemata solvuntur.

PROBLEMA 1

17. *Si curva EMN* (Fig. 2, p. 15) *ad axem APQ ita construatur, ut eius applicata quaeque PM aequalis sit quadranti AF ellipsis, cuius semiaxium coniugatorum alter sit ipsa abscissa AP, alter vero constans AE seu PF, invenire aequationem inter abscissam AP et applicatam PM naturam huius curvae exprimentem.*

SOLUTIO

Perspicuum est curvam EMN transire per punctum E, quoniam evanescente semiaxe ellipsis AP quadrans ellipsis abit in alterum semiaxem constantem AE. Recta porro AT ad angulum semirectum cum AP inclinata erit asymtotos curvae EMN, quia posito semiaxe AP infinite magno quadrans ellipticus huic ipsi semiaxi fit aequalis. Ad aequationem autem inveniendam sit $AE = c$, $AP = t$ et $PM = AF = z$, atque cum abscissa AP respectu ellipsis AF sit aequalis semiaxi eius, erit haec quaestio casus specialis aequationis inventae, quo est $t = a$ seu $x = 1$. Posito ergo $x = 1$ abibit superior aequatio in hanc

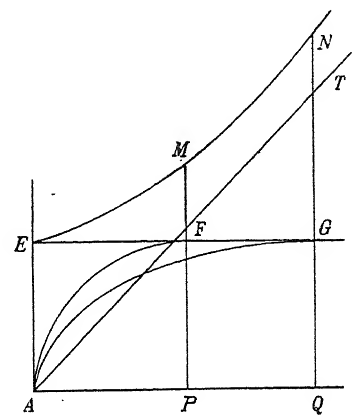

Fig. 2.

$$\frac{z}{bb+cc} = \frac{dz}{bdb} + \frac{1}{db}\,d.\frac{dz}{db}.$$

At quia est $bb = a^2 - c^2 = t^2 - c^2$, erit $bdb = tdt$ et $db = \frac{tdt}{\sqrt{(tt-c^2)}}$. Atque posito dt constante erit $ddb = -\frac{ccdt^2}{(tt-cc)^{\frac{3}{2}}}$. Hinc ergo fit

$$d.\frac{dz}{db} = \frac{ddz\sqrt{(tt-cc)}}{tdt} + \frac{ccdz}{tt\sqrt{(tt-cc)}},$$

unde oritur haec aequatio

$$\frac{z}{tt} = \frac{dz}{tdt} + \frac{ddz(tt-cc)}{ttdt^2} + \frac{ccdz}{t^3dt}$$

seu

$$tzdt^2 = (tt+cc)dtdz + tddz(tt-cc),$$

quae est aequatio quaesita pro curva proposita. Q. E. I.

18. Aequationem hanc sequenti modo ad differentialia primi gradus reduco ponendo $z = e^{\int sdt}$ existente $le = 1$; erit ergo

$$dz = e^{\int sdt}sdt \quad \text{et} \quad ddz = e^{\int sdt}(dsdt + ssdt^2).$$

Quibus valoribus substituendis oritur sequens aequatio

$$tdt = (t^2+c^2)sdt + t(tt-cc)ds + ts^2(t^2-c^2)dt;$$

quae ita est comparata, ut nullis adhuc cognitis artificiis indeterminatae a se invicem separari possint. Interim vero constructio huius aequationis ope rectificationis ellipsis constat.

19. Ne vero cuiquam dubium oriatur, quod posito $t=0$ fieri debeat $z=c$, cum tamen superiores integrationes ita accipi debeant, ut posito $x=0$ fiat quoque $z=0$, monendum est, quod quidem in hoc casu, quo $z=c$, sit $t=0$; non vero est quoque $x=0$, quia est $x=\frac{t}{a}$ et $t=a$ ideoque $x=1$, ita ut in hoc casu nusquam sit $x=0$, propterea z uspiam evanescere debeat.

20. Quemadmodum in hoc problemate posuimus $t=a$, ita quoque quaecunque aequatio inter t et a et constantes potest accipi et curva EMN definiri ita, ut quaevis applicata PM aequalis sit respondenti arcui elliptico AF. Habebitur enim loco superioris aequationis haec aequatio

$$\frac{z}{tt}=\frac{(tt+cc)dz}{t^3dt}+\frac{(tt-cc)ddz}{ttdt^2}+T$$

denotante T eam ipsius t functionem, quae ex terminis aequationis generalis, in quibus non inest z, oritur, si loco x ponatur $\frac{t}{a}$ et loco b eius valor $\sqrt{(a^2-c^2)}$ atque loco a eius valor in t ex aequatione inter a et t assumta substituatur. Neque vero haec aequatio tractatu est difficilior quam praecedens, in qua terminus T deest; reduci enim potest haec aequatio ad illam, uti iam alibi[1]) ostendi.

PROBLEMA 2

21. *Datis infinitis ellipsibus AOF, ANG, AMH* (Fig. 3, p. 17), *quarum alter semiaxis AE sit constans, alter vero variabilis ut AI, AK et AL, invenire aequationem pro curva $BONMC$, quae ab his omnibus ellipsibus arcus aequales AO, AN, AM abscindat.*

SOLUTIO

Ducta ad axem AC quacunque applicata MP curvae quaesitae sit $AP=t$, $PM=u$ et $AE=c$; ellipsis vero AMH semiaxis variabilis AL sit $=a$ et arcus abscissus AM, qui est constantis quantitatis, sit $=f$. Positis

1) Vide notam 3 p. 9. A. K.

nunc $x=\frac{t}{a}$ et $b=\sqrt{(a^2-c^2)}$ erit $z=f$ et $u=c\sqrt{(2x-xx)}$. His igitur substitutis generalis aequatio inter z, x et b abit in hanc

$$\frac{f}{bb+cc}=\frac{cc(1-x)\sqrt{(2x-xx)}}{(bb+cc)\sqrt{[cc+bb(2x-xx)]}}-\frac{dx}{bdb}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}}$$
$$-\frac{2bdx}{db}\sqrt{\frac{2x-xx}{cc+bb(2x-xx)}}+\frac{ccdx^2(1-x)}{db^2(2x-xx)^{\frac{3}{2}}\sqrt{[cc+bb(2x-xx)]}}$$
$$-\frac{1}{db}\,d.\frac{dx}{db}\sqrt{\frac{cc+bb(2x-xx)}{2x-xx}}.$$

Quia vero est $2x-xx=\frac{u^2}{c^2}$, multiplicetur ubique per

$$\sqrt{[cc+bb(2x-xx)]}=\frac{\sqrt{(c^4+bbuu)}}{c}$$

et prodibit

$$\frac{f\sqrt{(c^4+bbuu)}}{a^2c}=\frac{cu(1-x)}{a^2}-\frac{c^3dx}{budb}-\frac{3budx}{cdb}+\frac{c^5dx^2(1-x)}{u^3db^2}-\frac{(c^4+bbuu)}{cudb}\,d.\frac{dx}{db}.$$

In hac aequatione si loco b substituatur $\frac{\sqrt{(tt-ccxx)}}{x}$ et propter $x=\frac{c-\sqrt{(cc-uu)}}{c}$ prodibit tandem aequatio differentialis secundi gradus inter t et u, nempe coordinatas curvae quaesitae. Q. E. I.

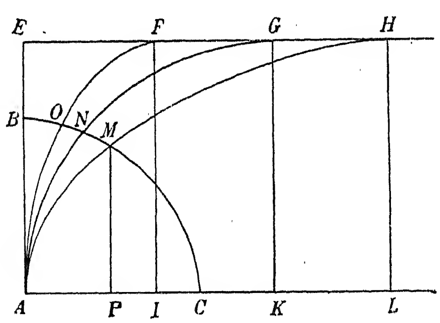

Fig. 3.

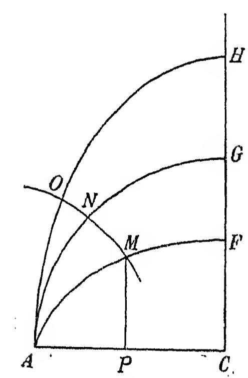

Fig. 4.

22. At si infinitae ellipses AMF, ANG et AOH (Fig. 4) omnes habeant axem horizontalem communem, ita ut C sit centrum omnium, pro hoc casu peculiarem aequationem modularem erui oportet, antequam curvam MNO definire licet, quae ab omnibus arcus aequales AM, AN, AO abscindat. Sit

igitur $AC = c$, $CF = a$, $AP = t$, $PM = u$ et arcus $AM = z$. His positis erit

$$u = \frac{a}{c}\sqrt{(2ct - tt)} \quad \text{et} \quad du = \frac{acdt - atdt}{c\sqrt{(2ct - tt)}}$$

ideoque fiet

$$z = \int \frac{dt}{c}\sqrt{\frac{a^2c^2 + (cc - aa)(2ct - tt)}{2ct - tt}} = \int \frac{du}{a}\sqrt{\frac{a^4 - a^2u^2 + ccuu}{aa - uu}}$$

atque posito $u = ay$ erit

$$z = \int dy \sqrt{\left(a^2 + \frac{ccyy}{1 - yy}\right)},$$

quod integrale ita debet accipi, ut z evanescat posito $y = 0$.

23. Si haec denuo differentietur posito praeter y et a variabili, habebitur

$$dz = dy\sqrt{\left(a^2 + \frac{ccyy}{1-yy}\right)} + da\int \frac{ady}{\sqrt{\left(a^2 + \frac{ccyy}{1-yy}\right)}}$$

atque posito

$$\frac{dz}{da} - \frac{dy}{da}\sqrt{\left(a^2 + \frac{ccyy}{1-yy}\right)} = R$$

erit

$$R = \int \frac{ady}{\sqrt{\left(a^2 + \frac{ccyy}{1-yy}\right)}}.$$

Hinc eodem modo fiet

$$dR = \frac{ady}{\sqrt{\left(a^2 + \frac{ccyy}{1-yy}\right)}} + da\int \frac{ccyydy}{(1 - yy)\left(a^2 + \frac{ccyy}{1-yy}\right)^{\frac{3}{2}}}$$

seu

$$\frac{dR}{da} - \frac{ady}{da\sqrt{\left(a^2 + \frac{ccyy}{1-yy}\right)}} = \int \frac{ccyydy}{(1 - yy)\left(aa + \frac{ccyy}{1-yy}\right)^{\frac{3}{2}}} = S$$

brevitatis gratia. Ponatur nunc $S + \alpha R + \beta z = Q$, ubi α et β sunt quantitates ab y liberae, Q vero functio ipsarum a et y, quae evanescat posito $y = 0$. Nunc ad α et β et Q invenienda differentietur haec aequatio posito a constante; erit

$$\frac{ccyydy\sqrt{(1-yy)}}{[aa(1-yy)+ccyy]^{\frac{3}{2}}}+\frac{\alpha ady\sqrt{(1-yy)}}{\sqrt{(a^2(1-yy)+ccyy)}}+\frac{\beta dy\sqrt{[a^2(1-yy)+ccyy]}}{\sqrt{(1-yy)}}$$
$$=\begin{pmatrix}ccyy-ccy^4+\alpha a^3-2\alpha a^3y^2+\alpha a^3y^4+\alpha accyy-\alpha accy^4\\ +\beta a^4-2\beta a^4y^2+\beta a^4y^4+2\beta a^2c^2y^2-2\beta a^2c^2y^4+\beta c^4y^4\end{pmatrix}dy$$
$$:(a^2(1-yy)+ccyy)^{\frac{3}{2}}\sqrt{(1-yy)}=dQ.$$

24. Sit
$$Q=\frac{\gamma y\sqrt{(1-yy)}}{\sqrt{(a^2(1-yy)+ccyy)}}$$
sumtoque huius differentiali posito a constante et aequatis terminis homologis erit
$$\alpha a+\beta a^2=\gamma,\quad \beta cc=-\gamma \quad \text{et} \quad 1+\alpha a+2\beta a^2=0.$$
Ex his fit
$$\alpha=\frac{a^2+c^2}{a(a^2-cc)},\quad \beta=\frac{-1}{a^2-c^2}\quad \text{et}\quad \gamma=\frac{cc}{aa-cc}.$$

Hisque valoribus substitutis pervenietur tandem ad hanc aequationem modularem
$$\frac{z}{aa-cc}=\frac{(a^2+c^2)dz}{ada(a^2-c^2)}+\frac{1}{da}d.\frac{dz}{da}-\frac{ccy\sqrt{(1-yy)}}{(a^2-c^2)\sqrt{(a^2(1-yy)+ccyy)}}$$
$$-\frac{(a^2+c^2)dy\sqrt{(a^2(1-yy)+ccyy)}}{a(a^2-c^2)da\sqrt{(1-yy)}}-\frac{2ady\sqrt{(1-yy)}}{da\sqrt{(a^2(1-yy)+ccyy)}}$$
$$-\frac{ccydy^2}{da^2(1-yy)^{\frac{3}{2}}\sqrt{(a^2(1-yy)+ccyy)}}-\frac{1}{da}d.\frac{dy}{da}\sqrt{\frac{a^2(1-yy)+ccyy}{1-yy}},$$
in qua a aeque sumtum est variabile ac y et z estque $y=\frac{u}{a}$.

25. Si nunc ex infinitis ellipsibus, quarum omnium alter axis est constans $2c$, alter variabilis $2a$, construatur curva EMN (Fig. 2, p. 15) hac lege, ut cuicunque abscissae $AP=a$ respondeat applicata PM, quae aequalis est quadranti elliptico sub semiaxibus a et c, hoc ergo casu erit $u=a$ et $y=1$ atque $PM=z$. Quare posito da constanti habebitur pro curva EMN haec aequatio
$$azda^2=(a^2+c^2)dadz+a(aa-cc)ddz.$$

Quae aequatio est ea ipsa, quam in solutione problematis 1 (§ 17) invenimus; congruit enim hic casus cum illo problemate atque, quod ibi erat t, hic est a.

PROBLEMA 3

26. *Descriptis infinitis ellipsibus AMF, ANG, AOH* (Fig. 4, p. 17) *commune centrum C communemque verticem A habentibus invenire curvam MNO, quae ab his omnibus ellipsibus arcus aequales AM, AN, AO abscindat.*

SOLUTIO

Posito omnium harum ellipsium semiaxe constante $AC = c$ ellipsisque cuiusvis AMF semiaxe altero variabili $CF = a$ atque curvae MNO abscissa $AP = t$ et applicata $PM = u$ fiat $\frac{u}{a} = y$ sitque longitudo $= f$, cui omnes arcus AM, AN, AO aequales sumantur. His positis et cum antecedentibus collatis erit $z = f$ ideoque

$$\frac{f}{a^2 - c^2} + \frac{ccy\sqrt{(1-yy)}}{(a^2 - c^2)\sqrt{(a^2(1-yy) + ccyy)}} + \frac{(a^2 + c^2)dy\sqrt{(a^2(1-yy) + ccyy)}}{a(a^2 - c^2)da\sqrt{(1-yy)}}$$
$$+ \frac{2ady\sqrt{(1-yy)}}{da\sqrt{(a^2(1-yy) + ccyy)}} + \frac{ccydy^2}{da^2(1-y^2)^{\frac{3}{2}}\sqrt{(a^2(1-yy) + ccyy)}}$$
$$+ \frac{1}{da}d.\frac{dy}{da}\sqrt{\frac{a^2(1-yy) + ccyy}{1-yy}} = 0$$

seu

$$\frac{f\sqrt{(1-yy)}}{(a^2 - c^2)\sqrt{(a^2(1-yy) + ccyy)}} + \frac{ccy(1-yy)}{(a^2 - c^2)(a^2(1-yy) + ccyy)} + \frac{(a^2 + c^2)dy}{a(a^2 - c^2)da}$$
$$+ \frac{2ady(1-yy)}{da(a^2(1-yy) + ccyy)} + \frac{ccydy^2}{da^2(1-yy)(a^2(1-yy) + ccyy)} + \frac{1}{da}d.\frac{dy}{da} = 0.$$

In qua aequatione si loco a ponatur $\frac{u}{y}$ et deinde loco y hic valor $\frac{\sqrt{(2ct - tt)}}{c}$, prodibit aequatio inter coordinatas t et u curvae quaesitae. Q. E. I.

ANIMADVERSIONES IN RECTIFICATIONEM ELLIPSIS

Commentatio 154 indicis ENESTROEMIANI
Opuscula varii argumenti 2, 1750, p. 121—166

1. Ellipsis rectificatio tot iam variis methodis est frustra tentata, ut non solum comparationem arcuum ellipticorum cum lineis rectis, sed etiam ne cum circularibus quidem aut parabolicis expectare nequeamus. Cum enim formula illa differentialis, cuius integrale arcum ellipticum indefinitum exprimit, nullo modo ab irrationalitate liberari queat, certum hoc est signum eius integrationem non solum non algebraice, sed etiam ne concessis quidem circuli et hyperbolae quadraturis perfici posse. Quod cum tenendum sit de rectificatione ellipsis indefinita, hinc adhuc minime sequitur arcum quempiam definitum veluti totam perimetrum ellipsis omnem comparationem cum lineis vel rectis vel circularibus penitus respuere, propterea quod iam innumerabiles curvae assignari possunt indefinite aeque parum rectificabiles atque ellipsis, in quibus tamen arcus definiti per lineas rectas mensurari queant.

2. Missa igitur rectificatione ellipsis indefinita definitam potius sum aggressus, experturus, utrum tota cuiusque ellipsis perimeter non commode possit ad mensuras cognitas, quorsum etiam logarithmos et arcus circulares refero, per expressiones finitas revocari. Quanquam autem in hac investigatione nihil admodum sum consecutus, quod scopo meo satisfecisset, tamen praeter expectationem nonnulla se mihi obtulerunt phaenomena satis singularia, quibus theoria linearum curvarum non mediocriter promoveri videtur. Tum vero etiam difficultates, quae in toto hoc calculo occurrerunt, ansam mihi praebuerunt quaedam insignia artificia inveniendi, quae tam in calculo integrali quam in theoria serierum infinitarum ingentem utilitatem saepius afferre posse videntur. Quamobrem operae pretium fore existimavi, si has speculationes totumque quasi filum calculorum meorum dilucide exposuero.

PROPOSITIO

3. *Super data recta* AC (Fig. 1) *tanquam altero semiaxe descriptos concipio infinitos quadrantes ellipticos* AP, AB, Ap, *quorum ergo omnium est centrum* C, *alteri vero semiaxes coniugati sunt* CP, CB, Cp. *Tum ex singulis punctis* P, B, p *arcus elliptici* PA, BA, pA *in directum extendantur, ita ut quaelibet* PQ *sit rectae* CA *parallela et quadranti elliptico* PYA *aequalis; quod si ubique fieri concipiatur, puncta haec* Q *sita erunt in linea quadam curva* $AQDq$, *cuius naturam investigare constitui.*

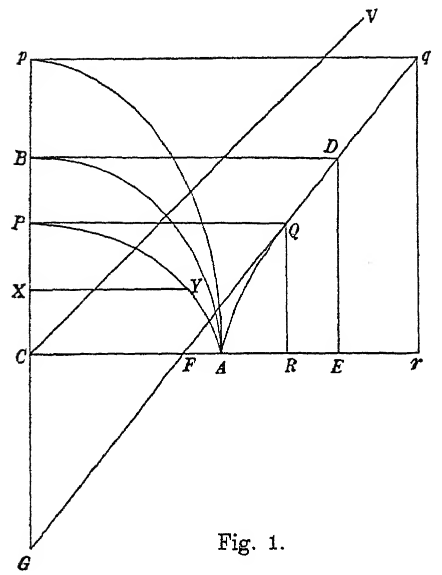

Fig. 1.

Ad genesin huius curvae vel leviter attendenti mox patebit, eam sequentes habere proprietates, quas evolvam, antequam in ipsam huius curvae indolem diligentius inquiram, ut eius figura et ductus saltem obiter perspiciatur.

4. Primum igitur, si in recta indefinita CBp, quae ad datam CA est normalis, capiatur quaevis abscissa CP, applicata PQ, quae ei respondet, erit aequalis quadranti perimetri ellipsis, cuius semiaxes coniugati sunt recta data CA et ipsa abscissa CP. Hinc si capiatur abscissa $CB = CA$, quo casu quadrans ellipticus abibit in quadrantem circularem AB, applicata respondens BD aequalis erit quartae parti peripheriae circuli radio AC descripti. Unde, si ratio diametri ad peripheriam ponatur $= 1 : \pi$, erit ista applicata $BD = \frac{1}{2}\pi \cdot AC$, sive ob $\pi = 3{,}1415926535897932$ erit

$$BD = 1{,}5707963267948966 . AC.$$

5. Secundo: Si abscissa CP evanescat, ellipsis evadet infinite angusta atque cum linea recta confundetur. Hoc ergo casu quadrans ellipticus abibit in ipsam lineam AC, cui propterea applicata abscissae evanescenti respondens erit aequalis. Quare ipsa recta CA erit applicata puncto C respondens et

curva quaesita per punctum A transibit. Huius ergo curvae iam duo habemus puncta cognita A et D, quorum alterum A geometrice datur, alterum vero D per rationem diametri ad peripheriam definitur.

6. Tertio: Ex cognito quovis curvae puncto Q intra A et D sito semper aliud quoddam curvae punctum q ultra D situm definiri potest. Capiatur enim Cp tertia proportionalis ad CP et CA, ut sit $Cp = \frac{CA \cdot CA}{CP}$; quia est $CP : CA = CA : Cp$, erit quadrans ellipticus Ap similis quadranti elliptico AP, cum utrinque eadem sit ratio inter semiaxes coniugatos. Hinc erit arcus Ap ad arcum AP ut AC ad CP ideoque $pq : PQ = AC : CP$ seu $pq = \frac{AC \cdot PQ}{CP}$. Consequenter si curvae quaesitae arcus AD tantum iam fuerit descriptus, ex eo reliqua curvae pars Dq in infinitum extensa definietur.

7. Quarto: Hinc iam insignis proprietas aequationis, qua natura curvae $AQDq$ exprimitur, agnoscitur. Si enim recta data AC unitate designetur, ut sit $AC = 1$, abscissa autem quaevis unitate minor $CP = p$ eique respondens applicata $PQ = q$, tum vero ponatur abscissa illa altera $Cp = P$ et applicata $pq = Q$, erit $P = \frac{1}{p}$ et $Q = \frac{q}{p}$. Quare cum inter P et Q eadem esse debeat aequatio, quae est inter p et q, patet aequationem inter p et q nullam mutationem esse subituram, si in ea loco p ubique scribatur $\frac{1}{p}$ et $\frac{q}{p}$ loco q. Unde, qualis ipsius p functio sit q, coniicere licet.

8. Quinto: Patet crescentibus abscissis CP applicatas continuo crescere, cum semper sint maiores quam abscissae. Verum si abscissae statuantur infinitae, applicatae ipsis fient aequales; discrimen enim prodibit infinite parvum, unde colligimus quaesitam curvam habere asymtotam et quidem rectam CV angulum rectum ACB bisecantem. Forma igitur huius curvae similis erit hyperbolae aequilaterae centrum in C, axem CA et asymtotam CV habentis. Ex descriptione porro intelligitur curvam infra rectam CA productam sui similem fore ideoque rectam CA eius fore diametrum perinde atque hyperbolae. Verumtamen hoc facile perspicitur nostram curvam multo lentius ad asymtotam suam CV appropinquare quam hyperbolam. Nam in hyperbola aequilatera, cui nostram curvam comparamus, quaevis applicata PQ aequalis est rectae lineae AP; unde, cum applicata nostrae curvae arcui AP sit aequalis, patet hyperbolam nostrae curvae fore circumscriptam, ita tamen, ut in initio A et in spatio infinito se mutuo tangant.

9. His affectionibus latius patentibus in genere notatis in ipsam huius curvae naturam accuratius inquiramus ac proposita quacunque abscissa $CP = p$ valorem respondentis applicatae $PQ = q$ investigemus; qui cum expressione finita contineri nequeat, per seriem infinitam exhiberi debebit. Sequens igitur resolvi oportet

PROBLEMA

10. *Ex datis semiaxibus CA et CP quadrantis elliptici CAP per seriem infinitam definire longitudinem arcus quadrantis AYP.*

SOLUTIO

Cum vocatus sit alter semiaxis $AC = 1$, alter vero $CP = p$ et arcus $AYP = q$, quaeratur primo arcus quivis indefinitus PY, qui vocetur $= s$. Iam ducta ad CP applicata normali YX sit $CX = x$ et $XY = y$; erit ex natura ellipsis $x = p\sqrt{(1 - yy)}$ hincque $dx = \frac{-pydy}{\sqrt{(1-yy)}}$. Fiet ergo ob $ds = \sqrt{(dx^2 + dy^2)}$

$$ds = \frac{dy\sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}},$$

unde integrando erit arcus

$$s = \int \frac{dy\sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}},$$

quae integratio ita institui debet, ut posito $y = 0$ fiat quoque $s = 0$, quia evanescente applicata $XY = y$ simul $PY = s$ evanescit. Hoc igitur integrali invento si ponatur $y = CA = 1$, arcus indefinitus PY abibit in longitudinem quadrantis elliptici $PYA = q$, quem quaerimus, ita ut sit

$$q = \int \frac{dy\sqrt{(1 - yy + ppyy)}}{\sqrt{(1 - yy)}},$$

siquidem peracta integratione ponatur $y = 1$.

11. Ad institutum ergo nostrum non est necesse, ut quaeramus valorem integralis huius indefiniti, sed eum tantum, quem induit, si post integrationem variabili y tribuatur valor determinatus $= 1$; quo pacto series multo simpli-

cior valorem q exprimens obtineri poterit. Ponatur enim brevitatis gratia $1-pp=nn$, ut sit $\sqrt{(1-yy+ppyy)}=\sqrt{(1-nnyy)}$, eritque hanc formulam in seriem evolvendo

$$\sqrt{(1-nnyy)}=1-\frac{1}{2}nnyy-\frac{1\cdot 1}{2\cdot 4}n^4y^4-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}n^6y^6-\text{etc.}$$

Quo valore substituto pro $\sqrt{(1-yy+ppyy)}$ arcus q ita exprimetur, ut sit

$$q=\int\frac{dy}{\sqrt{(1-yy)}}-\frac{1}{2}nn\int\frac{yy\,dy}{\sqrt{(1-yy)}}-\frac{1\cdot 1}{2\cdot 4}n^4\int\frac{y^4dy}{\sqrt{(1-yy)}}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}n^6\int\frac{y^6dy}{\sqrt{(1-yy)}}\text{ etc.,}$$

siquidem in singulis his integralibus post integrationem ponatur $y=1$.

12. Evolvamus ergo singula haec integralia; ac primo quidem ex circulo manifestum est formulam

$$\int\frac{dy}{\sqrt{(1-yy)}}$$

exprimere arcum circuli, cuius sinus $=y$ pro radio $=1$; unde posito $y=1$ haec formula dabit quartam peripheriae partem, cuius radius $=1$. Ideoque posita ratione diametri ad peripheriam $=1:\pi$ erit

$$\int\frac{dy}{\sqrt{(1-yy)}}=\frac{\pi}{2}$$

sicque iam adepti sumus valorem primi termini in serie nostra ante inventa.

13. Reliqui termini pari modo per valorem π commode poterunt exprimi; cuiusvis enim termini integratio ad integrationem praecedentis reducitur; quod quo facilius intelligatur, consideremus formulam quamcunque $\int\frac{y^\mu dy}{\sqrt{(1-yy)}}$; erit sequens $\int\frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}$. Iam assumamus hanc formulam algebraicam $y^{\mu+1}\sqrt{(1-yy)}$; cuius differentiale cum sit

$$\frac{(\mu+1)y^\mu dy-(\mu+2)y^{\mu+2}dy}{\sqrt{(1-yy)}},$$

erit vicissim

$$(\mu+1)\int\frac{y^\mu dy}{\sqrt{(1-yy)}}-(\mu+2)\int\frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}=y^{\mu+1}\sqrt{(1-yy)},$$

unde colligimus fore

$$\int\frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}=\frac{\mu+1}{\mu+2}\int\frac{y^\mu dy}{\sqrt{(1-yy)}}-\frac{1}{\mu+2}y^{\mu+1}\sqrt{(1-yy)}.$$

Quare invento integrali $\int\frac{y^\mu dy}{\sqrt{(1-yy)}}$ ex eo facile elicitur integrale sequens $\int\frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}$.

14. Quoniam vero eos tantum horum integralium valores desideramus, qui prodeunt posito $y=1$, hoc casu quantitas algebraica

$$\frac{1}{\mu+2}y^{\mu+1}\sqrt{(1-yy)}$$

evanescit eritque generatim pro casu $y=1$

$$\int\frac{y^{\mu+2}dy}{\sqrt{(1-yy)}}=\frac{\mu+1}{\mu+2}\int\frac{y^\mu dy}{\sqrt{(1-yy)}}.$$

Substituamus iam pro μ successive valores 0, 2, 4, 6, 8 etc., et quoniam vidimus esse

$$\int\frac{dy}{\sqrt{(1-yy)}}=\frac{\pi}{2},$$

erit, ut sequitur, si

$$\begin{aligned}
\mu&=0, & \int\frac{y^2dy}{\sqrt{(1-yy)}}&=\frac{1}{2}\int\frac{dy}{\sqrt{(1-yy)}}=\frac{1}{2}\cdot\frac{\pi}{2}\\
\mu&=2, & \int\frac{y^4dy}{\sqrt{(1-yy)}}&=\frac{3}{4}\int\frac{y^2dy}{\sqrt{(1-yy)}}=\frac{1\cdot3}{2\cdot4}\cdot\frac{\pi}{2}\\
\mu&=4, & \int\frac{y^6dy}{\sqrt{(1-yy)}}&=\frac{5}{6}\int\frac{y^4dy}{\sqrt{(1-yy)}}=\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{\pi}{2}\\
\mu&=6, & \int\frac{y^8dy}{\sqrt{(1-yy)}}&=\frac{7}{8}\int\frac{y^6dy}{\sqrt{(1-yy)}}=\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{\pi}{2},
\end{aligned}$$

unde lex, qua sequentes progrediuntur, sponte elucet.

15. Quodsi iam isti valores pro formulis integralibus, ex quibus longitudo quadrantis elliptici q conflari inventa est, substituantur, reperietur

$$q = \frac{\pi}{2} - \frac{1}{2}nn\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1\cdot 1}{2\cdot 4}n^4\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}n^6\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\quad \text{etc.},$$

quae ad sequentem seriem satis concinnam revocatur

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2}n^2 - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^4 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 - \text{etc.}\right),$$

cuius lex progressionis est manifesta. Restituatur ergo pro nn suus valor $1-pp$ eritque

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2}(1-pp) - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}(1-pp)^2 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}(1-pp)^3 - \text{etc.}\right).$$

16. Cum pro curva nostra $AQDq$ littera p exhibeat abscissam CP et littera q applicatam PQ, iam adepti sumus pro ista curva aequationem inter eius coordinatas p et q, quae, etsi serie constat infinita, tamen non solum eius naturam in se complectitur, sed etiam valores applicatae q mox satis accurate exhibet, si abscissa p parum ab unitate differat; hoc est, cum sit $CB = CA = 1$, si punctum P ipsi B fuerit proximum; tum enim ob $1-pp = nn$ quantitatem valde parvam series inventa valde convergit.

17. Hinc igitur indolem nostrae curvae prope punctum D, hoc est eius directionem et curvaturam definire poterimus. Primo enim patet, uti iam vidimus, si $p = 1$, fore $q = \frac{\pi}{2}$, ita ut sumta abscissa $CB = 1$ sit applicata

$$BD = \frac{\pi}{2} = 1{,}5707963267948966.$$

Deinde ad positionem tangentis inveniendam quaeratur ratio differentialium $dq : dp$, quae per differentiationem reperitur

$$\frac{dq}{dp} = \frac{\pi}{2}p\left\{\begin{aligned}&\frac{1}{2} + \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}(1-pp) + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6}(1-pp)^2\\ &\qquad + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}(1-pp)^3 + \text{etc.}\end{aligned}\right\}$$

Posito iam $p=1$ fiet $\frac{dq}{dp}=\frac{\pi}{4}$. Unde, si DG sit tangens curvae in puncto D, cum sit $BD:BG=dq:dp$, erit $BG=\frac{dp}{dq}\cdot BD=\frac{4}{\pi}\cdot BD$ et ob $BD=\frac{\pi}{2}$ fiet $BG=2=2BC$ et $CG=BC$. Sicque hoc casu subtangens BG erit dupla abscissae BC, et cum anguli BGD tangens sit

$$=\frac{dq}{dp}=\frac{\pi}{4}=0{,}78539816,$$

erit angulus $BGD=38^{0},\ 8^{\mathrm{I}},\ 45^{\mathrm{II}},\ 41^{\mathrm{III}},\ 51^{\mathrm{IV}}$.

18. Ad radium osculi seu evolutae in puncto D definiendum, cum sit ob $\frac{dq}{dp}=\frac{\pi}{4}$ elementum curvae

$$\sqrt{(dp^2+dq^2)}=dp\sqrt{\left(1+\frac{\pi\pi}{16}\right)},$$

erit radius osculi

$$=\left(1+\frac{\pi\pi}{16}\right)^{3:2}dp^2:ddq.$$

At sumendis differentialibus secundis erit

$$\frac{ddq}{dp^2}=\frac{\pi}{2}\left(\frac{1}{2}+\frac{1\cdot1\cdot3}{2\cdot2\cdot4}(1-pp)+\frac{1\cdot1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot4\cdot6}(1-pp)^2+\text{etc.}\right)$$
$$-\frac{\pi}{2}pp\left(\frac{1\cdot3}{2\cdot4}+\frac{1\cdot3\cdot3\cdot5}{2\cdot2\cdot4\cdot6}(1-pp)+\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7}{2\cdot2\cdot4\cdot4\cdot6\cdot8}(1-pp)^2+\text{etc.}\right).$$

Posito ergo $p=1$ erit

$$\frac{ddq}{dp^2}=\frac{\pi}{2}\left(\frac{1}{2}-\frac{3}{8}\right)=\frac{\pi}{16}.$$

Unde in puncto curvae D erit radius evolutae

$$=\frac{16}{\pi}\left(1+\frac{\pi\pi}{16}\right)\sqrt{\left(1+\frac{\pi\pi}{16}\right)},$$

qui valor in numeris proxime reperitur $=10{,}470678.$ [1])

1) Editio princeps: 10,470672. Correxit A. K.

19. Potest hinc adhuc alia series inveniri, quae valorem applicatae $PQ = q$ exprimat. Consideretur enim illud alterum curvae punctum q, pro quo sit abscissa $Cp = P$ et applicata $pq = Q$; erit quoque ob $P > 1$

$$Q = \frac{\pi}{2}\left(1 + \frac{1\cdot 1}{2\cdot 2}(PP-1) - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}(PP-1)^2 + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}(PP-1)^3 - \text{etc.}\right).$$

Iam vero supra notavimus, si sit $P = \frac{1}{p}$, fore $Q = \frac{q}{p}$; quare his valoribus substitutis impetrabimus novam aequationem inter p et q, qua natura curvae pariter exprimetur,

$$q = \frac{\pi}{2}p\left(1 + \frac{1\cdot 1}{2\cdot 2}\frac{(1-pp)}{pp} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\frac{(1-pp)^2}{p^4} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\frac{(1-pp)^3}{p^6} - \text{etc.}\right);$$

quae si cum ante inventa combinetur, innumerabiles aliae novae aequationes obtineri poterunt. Veluti si prior per p multiplicata ab hac subtrahatur, prodibit

$$q - pq = \frac{\pi}{2}p\left(\frac{1\cdot 1}{2\cdot 2}\frac{(1-pp)(1+pp)}{pp} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\frac{(1-pp)^2(1-p^4)}{p^4} + \text{etc.}\right),$$

quae reducitur ad hanc

$$q = \frac{\pi}{4}(1+p)\left(\frac{1}{2}\frac{1+pp}{p} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{(1-p^4)(1-pp)}{p^3} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6\cdot 6}\frac{(1+p^6)(1-pp)^2}{p^5} - \text{etc.}\right),$$

vel cum series adhuc sit divisibilis per $\frac{1+pp}{2p}$, erit

$$q = \frac{\pi}{8}\frac{(1+p)(1+pp)}{p}\left\{\begin{array}{l} 1 - \frac{1\cdot 3}{4\cdot 4}\frac{(1-pp)}{pp}(1-pp) + \frac{1\cdot 3\cdot 3\cdot 5}{4\cdot 4\cdot 6\cdot 6}\frac{(1-pp+p^4)}{p^4}(1-pp)^2 \\ \qquad - \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}\frac{(1-pp+p^4-p^6)}{p^6}(1-pp)^3 + \text{etc.} \end{array}\right\}$$

20. Manifestum autem est has series parum subsidii afferre, si applicatas invenire velimus, quae longius a BD, quae abscissae $p = 1$ respondet, sint remotae; si enim pro p ponatur numerus vel valde magnus vel valde parvus, series inventa vel parum admodum convergit vel etiam divergit. Si enim inde longitudinem primae applicatae CA, quae abcissae $p = 0$ respondet, definire velimus, serie primum inventa uti conveniet, quia in reliquis termini evadunt infinite magni. Habebimus igitur pro hoc casu $p = 0$

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} - \text{etc.}\right),$$

quae tam lente convergit, ut, etiamsi plurimi termini actu colligerentur, tamen verus ipsius q valor, quem novimus esse $=1$, inde difficillime agnosci posset.

21. Quanquam autem nunc quidem novimus esse

$$1-\frac{1\cdot 1}{2\cdot 2}-\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\text{etc.}=\frac{2}{\pi},$$

tamen inventio summae huius seriei non parum ardua videtur, si a priori tentetur. Veritatem quidem ex formula, quam quondam WALLISIUS pro circuli quadratura dedit[1]), intelligere licet, si termini ab initio in unum colligantur; sic enim prodit

$$1-\frac{1\cdot 1}{2\cdot 2}=\frac{1\cdot 3}{2\cdot 2},$$

$$\frac{1\cdot 3}{2\cdot 2}-\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}=\frac{1\cdot 3\cdot(4\cdot 4-1\cdot 1)}{2\cdot 2\cdot 4\cdot 4}=\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4},$$

$$\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}=\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6},$$

unde valor seriei in infinitum continuatae erit

$$\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdot 13}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdot 14}\ \text{etc.};$$

quae expressio cum sit ipsa WALLISIANA, patet summam nostrae seriei esse $=\frac{2}{\pi}$. Interim tamen iuvabit tradere methodum hanc seriem aliasque similes a priori summandi.

PROBLEMA

22. *Invenire summam huius seriei infinitae*

$$1-\frac{1\cdot 1}{2\cdot 2}-\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\text{etc.},$$

cuius lex progressionis primo intuitu est manifesta.

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1) I. WALLIS (1616—1703), *Arithmetica infinitorum sive nova methodus inquirendi in curvilineorum quadraturam aliaque difficiliora Matheseos problemata*; *Opera*, T. 1, p. 355, inprimis p. 469. A. K.

SOLUTIO

Ponatur summa huius seriei, quae quaeritur, $= s$, ut sit

$$s = 1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$$

Iam eligatur series, cuius summa constat et cuius coefficientes iam in his terminis contineantur. Cuiusmodi est haec

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}xx + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \text{etc.}$$

Erit ergo per differentiale quodpiam dP multiplicando et integrando

$$\int\frac{dP}{\sqrt{(1-xx)}} = P + \frac{1}{2}\int xx dP + \frac{1\cdot 3}{2\cdot 4}\int x^4 dP + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int x^6 dP + \text{etc.}$$

Nunc differentiale hoc dP ita definiatur, ut, si post integrationem ponatur $x = 1$, fiat

$$\int xx dP = -\frac{1}{2}P$$

$$\int x^4 dP = +\frac{1}{4}\int xx dP = -\frac{1\cdot 1}{2\cdot 4}P$$

$$\int x^6 dP = +\frac{3}{6}\int x^4 dP = -\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}P$$

$$\int x^8 dP = +\frac{5}{8}\int x^6 dP = -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}P;$$

quo facto si hi valores substituantur, habebitur

$$\int\frac{dP}{\sqrt{(1-xx)}} = P\left(1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}\right)$$

ideoque

$$\int\frac{dP}{\sqrt{(1-xx)}} = Ps,$$

si quidem post integrationem statuatur $x = 1$.

23. Huc ergo res redit, ut quaeratur formula differentialis dP, ut superioribus conditionibus satisfiat seu ut in genere sit

$$\int x^{\mu+2} dP = \frac{\mu - 1}{\mu + 2}\int x^{\mu} dP,$$

si quidem post integrationem utramque ponatur $x=1$. Omissa igitur hac conditione sit

$$\int x^{\mu+2}dP=\frac{\mu-1}{\mu+2}\int x^{\mu}dP+\frac{Qx^{\mu+1}}{\mu+2},$$

ubi Q eiusmodi sit functio ipsius x, quae evanescat posito $x=1$. Capiantur ergo differentialia eritque per x^{μ} dividendo

$$xxdP=\frac{\mu-1}{\mu+2}\,dP+\frac{xdQ+(\mu+1)\,Qdx}{\mu+2}$$

seu

$$0=(\mu-1)dP-(\mu+2)xxdP+xdQ+(\mu+1)\,Qdx,$$

quae aequatio, cum locum habere debeat pro omni valore ipsius μ, resolvetur in has duas

$$0=dP-xxdP+Qdx$$
$$0=-dP-2xxdP+xdQ+Qdx,$$

unde fit

$$dP=\frac{-Qdx}{1-xx}=\frac{xdQ+Qdx}{1+2xx}$$

et

$$xdQ(1-xx)=-Qdx(2+xx).$$

Quare cum sit

$$\frac{dQ}{Q}=-\frac{dx(2+xx)}{x(1-xx)}=-\frac{2dx}{x}-\frac{3xdx}{1-xx},$$

erit

$$Q=-\frac{(1-xx)^{\frac{3}{2}}}{xx}\quad\text{et}\quad dP=\frac{dx}{xx}\sqrt{(1-xx)}.$$

24. Verum hic notandum est, etsi valor ipsius Q evanescat posito $x=1$, tamen casu $\mu=0$ quantitatem algebraicam $\frac{Qx^{\mu+1}}{\mu+2}$ non evanescere, si ponatur $x=0$; quae tamen conditio aeque est necessaria atque altera, ita ut hoc casu non sit $\int xxdP=-\frac{1}{2}P$. Cum autem reliquae formulae, quibus $\mu>0$, locum habeant, a formula $\int xxdP$ erit incipiendum eritque

$$\int x^4dP=\frac{1}{4}\int xxdP$$
$$\int x^6dP=\frac{3}{6}\int x^4dP=\frac{1\cdot 3}{4\cdot 6}\int xxdP$$
$$\int x^8dP=\frac{5}{8}\int x^6dP=\frac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\int xxdP$$
etc.,

unde habebitur

$$\int \frac{dP}{\sqrt{(1-xx)}} = P + \int xx\,dP\left(\frac{1}{2} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6\cdot 6} + \text{etc.}\right).$$

At est

$$\frac{1}{2} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6\cdot 6} + \text{etc.} = 2(1-s)$$

ideoque

$$\int \frac{dP}{\sqrt{(1-xx)}} = P + 2(1-s)\int xx\,dP.$$

At ob $dP = \frac{dx}{xx}\sqrt{(1-xx)}$ erit

$$P = C - \frac{\sqrt{(1-xx)}}{x} - \text{A}\sin x,$$

$$\int xx\,dP = \int dx\sqrt{(1-xx)} = \frac{1}{2}\text{A}\sin x + \frac{1}{2}x\sqrt{(1-xx)}$$

et

$$\int \frac{dP}{\sqrt{(1-xx)}} = D - \frac{1}{x},$$

ubi constantes C et D ita accipi debent, ut integralia haec evanescant posito $x=0$; quanquam autem utraque seorsim fit infinita, tamen coniunctae se mutuo destruent. Erit enim

$$\int \frac{dP}{\sqrt{(1-xx)}} - P = D - \frac{1}{x} - C + \frac{\sqrt{(1-xx)}}{x} + \text{A}\sin x;$$

quae ut evanescat posito $x=0$, debet esse $D=C$ ideoque posito iam $x=1$ fiet

$$\int \frac{dP}{\sqrt{(1-xx)}} - P = -1 + \frac{\pi}{2},$$

et quia eodem hoc casu est $\int xx\,dP = \frac{\pi}{4}$, prodibit

$$-1 + \frac{\pi}{2} = 2(1-s)\frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{2}s$$

hincque colligitur fore $\frac{\pi}{2}s = 1$ et $s = \frac{2}{\pi}$ seu

$$1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.} = \frac{2}{\pi},$$

uti ex rei natura iam conclusimus.

25. Quoniam igitur eruimus in ipso initio esse applicatam curvae $CA = 1$, indolem huius curvae prope punctum A indagemus seu in valorem applicatae q inquiramus, si abscissa p fuerit valde parva. In hunc finem ponamus iterum $1 - pp = nn$, et cum sit

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2}nn - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^4 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 - \text{etc.}\right)$$

et quia novimus fore proxime $q = 1$, addamus aequalitatem modo inventam

$$0 = 1 - \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2} - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}\right)$$

atque habebimus

$$q = 1 + \frac{\pi}{2}\left(\frac{1\cdot 1}{2\cdot 2}(1 - nn) + \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}(1 - n^4) + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}(1 - n^6) + \text{etc.}\right);$$

cuius seriei cum singuli termini sint per $1 - nn = pp$ divisibiles, reducetur haec expressio ad hanc

$$q = 1 + \frac{\pi}{8}pp\left\{\begin{aligned} &1 + \frac{1\cdot 3}{4\cdot 4}(1 + nn) + \frac{1\cdot 3\cdot 3\cdot 5}{4\cdot 4\cdot 6\cdot 6}(1 + nn + n^4) \\ &\quad + \frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}(1 + n^2 + n^4 + n^6) + \text{etc.} \end{aligned}\right\}$$

26. Quodsi in hac expressione singuli termini ad potestates ipsius n evolvantur, reperietur

$$q = 1 + \frac{\pi}{2}pp\left\{\begin{aligned} &+\frac{1\cdot 1}{2\cdot 2} + \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} + \text{etc.} \\ &+ n^2\left(\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} + \text{etc.}\right) \\ &+ n^4\left(\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} + \text{etc.}\right) \\ &+ n^6\left(\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8} + \text{etc.}\right) \\ &\qquad\qquad \text{etc.} \end{aligned}\right\}$$

At ex supra inventis habemus summam primae seriei

$$\frac{1\cdot 1}{2\cdot 2} + \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} + \text{etc.} = 1 - \frac{2}{\pi};$$

quae si primo termino multetur, prodibit secunda, quae est coefficiens ipsius nn, ita ut sit

$$\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}+\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}+\text{etc.}=\frac{1\cdot 3}{2\cdot 2}-\frac{2}{\pi};$$

haec denuo primo termino multata dabit coefficientem ipsius n^4, nempe

$$\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}+\text{etc.}=\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}-\frac{2}{\pi}$$

similique modo coefficiens ipsius n^6 erit

$$=\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\frac{2}{\pi}$$

et ita porro sicque tandem obtinebitur

$$q=1+\frac{\pi}{2}pp\left\{\begin{aligned}&\left(1-\frac{2}{\pi}\right)+\left(\frac{1\cdot 3}{2\cdot 2}-\frac{2}{\pi}\right)nn+\left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}-\frac{2}{\pi}\right)n^4\\&\qquad+\left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\frac{2}{\pi}\right)n^6+\text{etc.}\end{aligned}\right\}$$

vel erit

$$q=1+pp\left\{\begin{aligned}&\left(\frac{\pi}{2}-1\right)+\left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2}-1\right)nn+\left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\cdot\frac{\pi}{2}-1\right)n^4\\&\qquad+\left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot\frac{\pi}{2}-1\right)n^6+\text{etc.}\end{aligned}\right\}$$

27. Ponamus iam hic $n=1$, ut obtineamus aequationem huius formae $q=1+App$, qua natura curvae prope punctum A exprimitur; cum enim coniectare liceat veram aequationem futuram esse huius formae

$$q=1+App+Bp^4+Cp^6+Dp^8+\text{etc.},$$

si abscissa p valde parva assumatur, reliqui termini praeter binos primos ommitti poterunt atque ex aequatione $q=1+App$ tam positio tangentis quam curvatura in puncto A colligi poterit. Posito enim $AR=x$, $RQ=y$ erit $q=1+x$ et $p=y$ ideoque, si arcus AQ fuerit minimus, is cum parabola confundetur, cuius aequatio $x=Ayy$ seu $yy=\frac{1}{A}x$ ac propterea $\frac{1}{A}$ parameter. Unde sequitur tangentem curvae in A fore ad rectam AC perpendicularem et radium osculi ibidem esse $=\frac{1}{2A}$.

28. Hic igitur coefficiens A reperietur, si in superiore serie, per quam quantitas pp multiplicatur, ponatur $n=1$, ita ut sit

$$A=\left(\frac{\pi}{2}-1\right)+\left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2}-1\right)+\left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\cdot\frac{\pi}{2}-1\right)+\text{etc.},$$

quae autem, si eius summatio tentetur, tam parum convergens deprehenditur, ut eius summam adeo infinitam suspicari debeamus. In hac autem suspicione eo magis confirmamur, si seriem primo (§ 15) inventam secundum dimensiones ipsius p evolvamus, unde fit

$$q=\frac{\pi}{2}\left\{\begin{array}{l} 1-\frac{1\cdot 1}{2\cdot 2}-\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\text{etc.}\\ +pp\left(\frac{1\cdot 1}{2\cdot 2}+\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\cdot 2+\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot 3+\text{etc.}\right)\\ -p^4\left(\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}+\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot 3+\text{etc.}\right)\\ \qquad\qquad\text{etc.}\end{array}\right\}$$

29. Hinc ergo coefficiens ipsius pp in aequatione generali pro curva

$$q=1+App+Bp^4+Cp^6+Dp^8+\text{etc.}$$

erit

$$A=\frac{\pi}{2}\left(\frac{1\cdot 1}{2\cdot 2}\cdot 1+\frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}\cdot 2+\frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot 3+\text{etc.}\right)$$

seu

$$A=\frac{\pi}{4}\left(\frac{1}{2}+\frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}+\text{etc.}\right)$$

similique modo et reliquos coefficientes B, C, D etc. ex hac serie eruere licebit. Verum hoc labore supersedere poterimus, cum liqueat non solum coefficientem A, sed etiam omnes reliquos prodituros esse infinitos. Perspicuum hoc fiet ex solutione huius problematis.

PROBLEMA

30. *Invenire summam huius seriei infinitae*

$$s=\frac{1}{2}+\frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6}+\text{etc.}$$

SOLUTIO

Assumatur ad hanc summam s inveniendam haec formula

$$\frac{1}{\sqrt{(1-xx)}}=1+\frac{1}{2}xx+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\text{etc.},$$

ut sit

$$\int\frac{dP}{\sqrt{(1-xx)}}=P+\frac{1}{2}\int xxdP+\frac{1\cdot 3}{2\cdot 4}\int x^4dP+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\int x^6dP+\text{etc.},$$

sitque, si post integrationes singulas ponatur $x=1$,

$$\int xxdP=\frac{3}{4}P$$

$$\int x^4\,dP=\frac{5}{6}\int xxdP=\frac{3\cdot 5}{4\cdot 6}P$$

$$\int x^6\,dP=\frac{7}{8}\int x^4\,dP=\frac{3\cdot 5\cdot 7}{4\cdot 6\cdot 8}P$$

etc.

hincque fiet

$$\int\frac{dP}{\sqrt{(1-xx)}}=P\left(1+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6}+\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}+\text{etc.}\right)$$

sive

$$\int\frac{dP}{\sqrt{(1-xx)}}=2Ps;$$

unde invento P reperietur s, si post integrationem ponatur $x=1$.

31. Cum igitur generaliter esse debeat

$$\int x^{\mu+2}dP=\frac{\mu+3}{\mu+4}\int x^{\mu}dP+\frac{x^{\mu+1}Q}{\mu+4},$$

dummodo Q eiusmodi sit functio, quae evanescat posito $x=1$, erit

$$(\mu+4)xxdP=(\mu+3)dP+xdQ+(\mu+1)Qdx,$$

unde duae sequentes aequationes conficiuntur

$$xxdP=dP+Qdx$$

$$4xxdP=3dP+xdQ+Qdx$$

et

$$dP=\frac{-Qdx}{1-xx}=\frac{-xdQ-Qdx}{3-4xx}$$

hincque elicitur

$$\frac{dQ}{Q} = \frac{2dx - 3xxdx}{x(1-xx)} = \frac{2\,dx}{x} - \frac{xdx}{1-xx}$$

et

$$Q = -\,xx\sqrt{(1-xx)}.$$

Quare habebitur

$$dP = \frac{xxdx}{\sqrt{(1-xx)}} \quad \text{et} \quad \frac{dP}{\sqrt{(1-xx)}} = \frac{xxdx}{1-xx} = -\,dx + \frac{dx}{1-xx}.$$

Fiet ergo $P = \frac{1}{4}\pi$, si post integrationem ponatur $x = 1$, at

$$\int \frac{dP}{\sqrt{(1-xx)}} = -\,x + \frac{1}{2}\,l\,\frac{1+x}{1-x},$$

cuius valor posito $x = 1$ fit utique infinitus. Erit igitur $s = \infty$ seu summa seriei propositae infinite magna.

32. Quia igitur coefficiens A ipsius pp in aequatione

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + \text{etc.}$$

est infinitus, radius osculi curvae in puncto A utique erit infinite parvus. Verum praeterea haec aequatio, in qua omnes omnino coefficientes A, B, C, D etc. fiunt infiniti, nihil plane ad curvae cognitionem confert. Quia enim radius osculi curvae in A est infinite parvus, natura curvae circa punctum A huiusmodi aequatione $q = 1 + \alpha p^m$ exprimetur, in qua exponens m binario sit minor, verumtamen unitate maior; sed ex omnibus, quae hactenus sunt tradita, nulla via patet, qua hunc exponentem m scrutari queamus. Cum enim is numerus integer esse nequeat, nulla serierum, quas pro q eruimus, ita est comparata, ut ex ea potestatem ipsius p irrationalem elicere liceat.

33. Hinc intelligimus problema esse summopere difficile, quo aequatio tantum elementaris requiritur, quae naturam curvae propositae $AQDq$ saltem proxime circa punctum A exhibeat. Notum est enim, si ponatur $AR = x$ et

erit curva AQ, naturam minimae eius portiunculae circa

quatione $y^m = Ax$ comprehendi posse, siquidem curva

rvis autem transcendentibus certum videtur quasvis

culas cum arcubus curvarum algebraicarum comparari

posse. Quare in nostra curva, etsi est transcendens, hoc eo magis mirum videri debet, quod nulla huiusmodi formula $y^m = Ax$ exhiberi possit, quae saltem minimae eius portiunculae circa A sitae naturam declaret.

34. Hunc nodum ut resolvamus, aequationem nobis finitam inter coordinatas p et q investigare oportebit, quae etsi, ut facile praevidere licet, ad differentialia secundi ordinis exsurget, tamen ad accuratiorem curvae cognitionem magis erit accommodata. Eliciemus autem huiusmodi aequationem, quae numero terminorum finito constet, si seriem primo inventam ad summam revocabimus. Cum enim posito $1 - pp = nn$ sit

$$\frac{2q}{\pi} = 1 - \frac{1 \cdot 1}{2 \cdot 2} nn - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^4 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} n^6 - \text{etc.},$$

erit differentiando

$$\frac{2\,dq}{\pi\,dn} = -\frac{1 \cdot 1}{2} n - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4} n^3 - \frac{1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} n^5 - \text{etc.},$$

quae per n multiplicata denuoque differentiata dat

$$\frac{2}{\pi\,dn}\, d.\frac{n\,dq}{dn} = -1 \cdot 1 n - \frac{1 \cdot 1}{2 \cdot 2} \cdot 1 \cdot 3 n^3 - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \cdot 3 \cdot 5 n^5 - \text{etc.}$$

Multiplicetur haec per $\frac{dn}{n}$ ac rursus integretur; erit

$$\frac{2}{\pi} \int \frac{1}{n}\, d.\frac{n\,dq}{dn} = -1 n - \frac{1 \cdot 1}{2 \cdot 2} \cdot 1 n^3 - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} \cdot 3 n^5 - \text{etc.}$$

Multiplicetur per $\frac{dn}{n^3}$ et integrando prodibit

$$\frac{2}{\pi} \int \frac{dn}{n^3} \int \frac{1}{n}\, d.\frac{n\,dq}{dn} = \frac{1}{n} - \frac{1 \cdot 1}{2 \cdot 2} n - \frac{1 \cdot 1 \cdot 1 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4} n^3 - \text{etc.},$$

quae series cum sit ipsa proposita per n divisa, erit

$$\frac{2}{\pi} \int \frac{dn}{n^3} \int \frac{1}{n}\, d.\frac{n\,dq}{dn} = \frac{2q}{\pi n} \quad \text{seu} \quad \int \frac{dn}{n^3} \int \frac{1}{n}\, d.\frac{n\,dq}{dn} = \frac{q}{n}.$$

35. Sumamus nunc differentialia habebiturque

$$\frac{dn}{n^3} \int \frac{1}{n}\, d.\frac{n\,dq}{dn} = \frac{n\,dq - q\,dn}{nn} \quad \text{seu} \quad \int \frac{1}{n}\, d.\frac{n\,dq}{dn} = \frac{nn\,dq}{dn} - nq$$

porroque differentiando

$$\frac{1}{n}\, d.\frac{ndq}{dn} = nd.\frac{ndq}{dn} + ndq - ndq - qdn$$

seu

$$(1-nn)\, d.\frac{ndq}{dn} + qndn = 0.$$

Iam ob $1 - nn = pp$ erit

$$ndn = -pdp \quad \text{et} \quad \frac{dn}{n} = -\frac{pdp}{1-pp},$$

unde fit

$$-ppd.\frac{(1-pp)dq}{pdp} - pqdp = 0 \quad \text{seu} \quad d.\frac{(1-pp)dq}{pdp} + \frac{qdp}{p} = 0.$$

Sumatur iam dp constans; erit

$$\frac{(1-pp)ddq}{pdp} - \frac{dq(1+pp)}{pp} + \frac{qdp}{p} = 0$$

seu

$$p(1-pp)ddq - dpdq(1+pp) + pqdp^2 = 0.$$

36. En igitur aequationem differentialem secundi gradus pro curva proposita

$$p(1-pp)ddq - dpdq(1+pp) + pqdp^2 = 0,$$

ex qua potestas illa ipsius p in aequatione $q = 1 + Ap^m$ elici debet, si abscissa p valde parva statuatur. Cum igitur fiat

$$dq = mAp^{m-1}dp \quad \text{et} \quad ddq = m(m-1)Ap^{m-2}dp^2,$$

orietur

$$\left.\begin{array}{r} m(m-1)Ap^{m-1} - \quad mAp^{m-1} + p \\ -m(m-1)Ap^{m+1} - \quad mAp^{m+1} + Ap^{m+1} \end{array}\right\} = 0$$

seu

$$m(m-2)Ap^{m-1} - (mm-1)Ap^{m+1} + p = 0.$$

Deberet ergo esse $m = 2$, ut terminus Ap^{m-1} cum p comparari posset, sed [illegible]m obtinetur $A = \infty$; praeterea vero hinc perspicitur exponentem m [illegible] numerum fractum esse posse, ita ut hinc difficultas supra me- [illegible]ugeri potius quam tolli videatur.

37. Quodsi regulis consuetis uti velimus ad aequationem inventam in seriem evolvendam, quae secundum potestates ipsius p procedat, quoniam novimus primum seriei terminum esse $=1$, nullam aliam formam inde colligere licet nisi hanc

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.},$$

unde fit

$$\frac{dq}{dp} = 2Ap + 4Bp^3 + 6Cp^5 + 8Dp^7 + \text{etc.}$$

et

$$\frac{ddq}{dp^2} = 2A + 12Bp^2 + 30Cp^4 + 56Dp^6 + \text{etc.},$$

qui valores in aequatione substituti praebebunt

$$\left.\begin{array}{rrrrrr}
+\,2Ap & +\,12Bp^3 & +\,30Cp^5 & +\,56Dp^7 & +\,\text{etc.} \\
 & -\,2A & -\,12B & -\,30C & -\,\text{etc.} \\
-\,2A & -\,4B & -\,6C & -\,8D & -\,\text{etc.} \\
 & -\,2A & -\,4B & -\,6C & -\,\text{etc.} \\
+\,1 & +\,A & +\,B & +\,C & +\,\text{etc.}
\end{array}\right\} = 0,$$

unde omnes coefficientes A, B, C etc. prodeunt infiniti.

38. Hinc igitur videmus regulas ordinarias, secundum quas vulgo forma seriei, in quam aequatio differentialis transmutanda sit, diiudicari solet, non esse sufficientes, cum hoc casu nullam afferant utilitatem; unde nostra aequatio eo maiorem meretur attentionem. Sequenti tamen modo ex ea natura curvae prope punctum A colligi poterit, ex quo simul intelligetur, quemadmodum quoque in aliis casibus defectus iste regularum usu receptarum suppleri eaeque ad praxin accommodari debeant. Quia enim abscissam p hic pro infinite parva habemus, in aequatione pro $1-pp$ et $1+pp$ ponere licebit 1, et quia novimus esse hoc casu proxime $q=1$, pro quantitate finita q unitatem scribamus; quo facto aequatio differentio-differentialis inventa pro casu, quo abscissa p est minima, sequentem induet formam

$$pddq - dpdq + pdp^2 = 0.$$

39. Huius iam aequationis resolutio est facilis; cum enim dp sit constans, ponatur $dq = rdp$; erit $ddq = drdp$ habebiturque

$$pdr - rdp + pdp = 0$$

sive

$$\frac{pdr - rdp}{pp} + \frac{dp}{p} = 0,$$

cuius integrale est $\frac{r}{p} + lp = C$, unde fit $r = Cp - plp$ ideoque

$$dq = Cpdp - pdplp.$$

Haec iam aequatio integrata dabit

$$q = 1 + \frac{1}{2}Cp^2 - \frac{1}{2}pp\,lp + \frac{1}{4}pp,$$

in qua cum terminus pp incomparabiliter sit minor quam $pplp$, erit pro curvae initio A

$$q = 1 - \frac{1}{2}pplp.$$

40. Nunc igitur naturam curvae prope initium A aequatione simplici definire possumus; si enim vocemus $AR = x$ et $RQ = y$, ob $p = y$ et $q = 1 + x$ orietur haec $x = -\frac{1}{2}yyly$, ad quam aequatio generalis pro curva revocatur, si coordinatae x et y sint quam minimae. Patet igitur ne minimum quidem arculum circa A tanquam portiunculam curvae algebraicae spectari posse, sed eius naturam logarithmos implicare. Et quoniam aequatio logarithmica in exponentialem transformari potest, initium curvae nostrae A commune erit cum linea transcendente, cuius aequatio est $e^{-2x} = y^{yy}$ sumto e pro numero, cuius logarithmus hyperbolicus est $= 1$.

41. Aequatione hac $x = -\frac{1}{2}yyly$ confirmantur quoque ea, quae supra iam de affectionibus huius curvae in puncto A notavimus. Primo enim patet, si sit $y = 0$, fore quoque $yyly$ ac proinde $x = 0$, etsi hoc casu sit $ly = -\infty$. Deinde cum sit $dx = -ydyly - \frac{1}{2}ydy$, quia y incomparabiliter est minus quam yly, erit $dx = -ydyly$ ac propterea $\frac{dy}{dx} = \frac{-1}{yly} = \infty$ posito $y = 0$; unde patet tangentem curvae in A ad abscissam AR esse perpendicularem. Porro

cum sit subnormalis $\frac{ydy}{dx} = \frac{-1}{ly}$ hocque casu subnormalis radio evolutae aequetur, ob $ly = \infty$, si $y = 0$, manifestum est radium osculi curvae in A esse infinite parvum.

42. Maxime autem differt haec curva a curvis algebraicis, quae in initio A quoque habent radium osculi evanescentem. Curvarum enim algebraicarum, quae hac indole gaudent, natura circa initium A huiusmodi formula exprimitur $x = \alpha y^m$ existente $m < 2$, attamen $m > 1$. Sit igitur $m = 2 - \omega$ existente ω fractione unitate minore, ut sit $x = \alpha y^{2-\omega}$; erit $dx = \alpha(2-\omega)y^{1-\omega}dy$ ideoque

$$\frac{dy}{dx} = \frac{1}{\alpha(2-\omega)y^{1-\omega}} = \infty$$

ob $y^{1-\omega} = 0$; at radius osculi, qui subnormali $\frac{ydy}{dx}$ aequalis est, erit $= \frac{y^\omega}{\alpha(2-\omega)} = 0$. Pro nostra vero curva radius osculi inventus est $= \frac{-1}{ly}$, unde radius osculi evanescens in curva algebraica quacunque erit ad radium osculi in nostrae curvae puncto A ut $-y^\omega ly$ ad $\alpha(2-\omega)$, hoc est ut 0 ad 1; quantumvis enim exiguus sit exponens ω, casu $y = 0$ semper est $y^\omega ly = 0$, etiamsi sit $ly = -\infty$. Quare in nostra quidem curva radius osculi in A est infinite parvus, sed tamen infinities maior est quam radius osculi evanescens in omni curva algebraica.

43. Cognito iam initio seriei, qua valor applicatae $PQ = q$ per abscissam $CP = p$ exprimitur, scilicet

$$q = 1 - \frac{1}{2}pplp + App,$$

non difficile erit hinc formam totius seriei colligere. Cum enim ex aequatione differentio-differentiali intelligatur sequentium terminorum potestates ipsius p binario crescere, valor ipsius q generatim gemina serie infinita exprimetur eritque

$$\begin{aligned} q = 1 &+ Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.} \\ &- \alpha pplp - \beta p^4lp - \gamma p^6lp - \delta p^8lp - \text{etc.}, \end{aligned}$$

in qua quidem nunc iam novimus esse $\alpha = \frac{1}{2}$.

44. Cum igitur verus valor ipsius q duplici serie contineatur, ut utramque seorsim eliciamus, ponamus

$$q = r - slp$$

eritque differentiando

$$dq = dr - \frac{sdp}{p} - dslp, \quad ddq = ddr - \frac{2\,dpds}{p} + \frac{sdp^2}{pp} - ddslp.$$

Hi valores in nostra aequatione differentiali

$$p(1-pp)ddq - dpdq(1+pp) + pqdp^2 = 0$$

substituantur ac termini per lp affecti seorsim nihilo aequentur; hoc modo duae obtinebuntur aequationes

$$\text{I.}\quad p(1-pp)dds - (1+pp)dpds + psdp^2 = 0,$$

$$\text{II.}\quad p(1-pp)ddr - (1+pp)dpdr + prdp^2 - 2(1-pp)dpds + \frac{2sdp^2}{p} = 0.$$

45. Ad has aequationes resolvendas ponatur

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.}$$
$$s = \alpha p^2 + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \text{etc.}$$

eritque differentialibus sumendis

$$\frac{dr}{dp} = 2Ap + 4Bp^3 + 6\,Cp^5 + 8\,Dp^7 + \text{etc.}$$

$$\frac{ddr}{dp^2} = 2A + 12\,Bp^2 + 30\,Cp^4 + 56\,Dp^6 + \text{etc.}$$

$$\frac{ds}{dp} = 2\,\alpha p + 4\beta p^3 + 6\gamma p^5 + 8\,\delta p^7 + \text{etc.}$$

$$\frac{dds}{dp^2} = 2\alpha + 12\beta p^2 + 30\gamma p^4 + 56\,\delta p^6 + \text{etc.}$$

His valoribus substitutis prima aequatio abibit in hanc

$$\left.\begin{array}{rrrrrrr} & 2\alpha p + & 12\beta p^3 + & 30\gamma p^5 + & 56\delta p^7 + & 90\varepsilon p^9 + & \text{etc.} \\ & -\ 2\alpha & -\ 12\beta & -\ 30\gamma & -\ 56\delta & & -\ \text{etc.} \\ -\ 2\alpha & -\ 4\beta & -\ 6\gamma & -\ 8\delta & -\ 10\varepsilon & & -\ \text{etc.} \\ & -\ 2\alpha & -\ 4\beta & -\ 6\gamma & -\ 8\delta & & -\ \text{etc.} \\ & +\ \alpha & +\ \beta & +\ \gamma & +\ \delta & & +\ \text{etc.} \end{array}\right\} = 0.$$

46. Si iam singularum potestatum ipsius p coefficientes nihilo aequales ponantur, erit

$$
\begin{array}{ll}
2\alpha - 2\alpha = 0; & \alpha \text{ manet indeterminatum} \\[2ex]
8\beta - 3\alpha = 0; & \beta = \dfrac{1\cdot 3}{2\cdot 4}\alpha \\[2ex]
24\gamma - 15\beta = 0; & \gamma = \dfrac{3\cdot 5}{4\cdot 6}\beta = \dfrac{1\cdot 3\cdot 3\cdot 5}{2\cdot 4\cdot 4\cdot 6}\alpha \\[2ex]
48\delta - 35\gamma = 0; & \delta = \dfrac{5\cdot 7}{6\cdot 8}\gamma = \dfrac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}\alpha \\[2ex]
80\varepsilon - 63\delta = 0; & \varepsilon = \dfrac{7\cdot 9}{8\cdot 10}\delta = \dfrac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10}\alpha \\[2ex]
\text{etc.} & \text{etc.}
\end{array}
$$

Si igitur valor coefficientis primi α constaret, quem quidem iam vidimus esse $=\frac{1}{2}$, omnes sequentes coefficientes β, γ, δ etc. forent cogniti. Verum resolutio alterius aequationis quoque hunc nobis valorem ipsius α patefaciet.

47. Substitutis enim seriebus ante traditis in altera aequatione proveniet

$$
\left.
\begin{array}{rrrrrr}
2Ap & +12Bp^3 & +30Cp^5 & +56Dp^7 & +90Ep^9 & +\text{etc.} \\
 & -2A & -12B & -30C & -56D & -\text{etc.} \\
-2A & -4B & -6C & -8D & -10E & -\text{etc.} \\
 & -2A & -4B & -6C & -8D & -\text{etc.} \\
+1 & +A & +B & +C & +D & +\text{etc.} \\
-4\alpha & -8\beta & -12\gamma & -16\delta & -20\varepsilon & -\text{etc.} \\
 & +4\alpha & +8\beta & +12\gamma & +16\delta & +\text{etc.} \\
+2\alpha & +2\beta & +2\gamma & +2\delta & +2\varepsilon & +\text{etc.}
\end{array}
\right\} = 0.
$$

Unde simili modo elicitur

$$2A - 2A + 1 - 2\alpha = 0; \qquad \text{hinc fit} \quad \alpha = \frac{1}{2}$$

$$8B - 3A - 6\beta + 4\alpha = 0; \qquad 2\cdot 4\, B - 1\cdot 3A + 2\left(2 - \frac{1\cdot 3\cdot 3}{2\cdot 4}\right)\alpha = 0$$

$$24C - 15B - 10\gamma + 8\beta = 0; \qquad 4\cdot 6\, C - 3\cdot 5B + 2\left(4 - \frac{3\cdot 5\cdot 5}{4\cdot 6}\right)\beta = 0$$

$$48D - 35C - 14\delta + 12\gamma = 0; \qquad 6\cdot 8\, D - 5\cdot 7C + 2\left(6 - \frac{5\cdot 7\cdot 7}{6\cdot 8}\right)\gamma = 0$$

$$80E - 63D - 18\varepsilon + 16\delta = 0; \qquad 8\cdot 10E - 7\cdot 9D + 2\left(8 - \frac{7\cdot 9\cdot 9}{8\cdot 10}\right)\delta = 0$$

etc. etc.

48. Cognito igitur valore ipsius $\alpha = \frac{1}{2}$ altera series s, quae logarithmum ipsius p involvit, tota innotescit; erit enim

$$\alpha = \frac{1}{2}$$

$$\beta = \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4}$$

$$\gamma = \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6}$$

$$\delta = \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}$$

$$\varepsilon = \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10}$$

etc.

fietque hinc

$$s = \alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \text{etc.}$$

49. Quod autem ad alteram seriem attinet

$$r = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \text{etc.},$$

primus coefficiens A hinc manet indeterminatus, cuius rei ratio est, quod has series ex aequatione differentiali secundi gradus elicuimus, quae duplici determinatione indiget, ut ad nostrum casum accommodetur. Quare valorem

huius coefficientis A ex ipsa curvae natura definiri oportet, eo autem invento reliqui innotescent ex his formulis, ad quas superiores redeunt:

$$B = \frac{1\cdot 3}{2\cdot 4}A - \frac{1}{8}\alpha\left(\frac{3}{2\cdot 2} + \frac{1}{1\cdot 1}\right)$$

$$C = \frac{3\cdot 5}{4\cdot 6}B - \frac{1}{8}\beta\left(\frac{3}{3\cdot 3} + \frac{1}{2\cdot 2}\right)$$

$$D = \frac{5\cdot 7}{6\cdot 8}C - \frac{1}{8}\gamma\left(\frac{3}{4\cdot 4} + \frac{1}{3\cdot 3}\right)$$

$$E = \frac{7\cdot 9}{8\cdot 10}D - \frac{1}{8}\delta\left(\frac{3}{5\cdot 5} + \frac{1}{4\cdot 4}\right)$$

etc.

50. His autem omnibus coefficientibus inventis ad datam quamvis abscissam $CP = p$ valor respondentis applicatae $PQ = q$ ita definitur, ut sit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + \text{etc.}$$
$$- \alpha pplp - \beta p^4lp - \gamma p^6lp - \delta p^8lp - \text{etc.},$$

quae series, si abscissa p fuerit unitate multo minor, satis promte convergit, ut inde valor ipsius q cognosci queat. Hinc vero etiam applicatae, quae abscissis multo maioribus unitate respondent, definiri poterunt, quia abscissae $\frac{1}{p}$ respondet applicata $\frac{q}{p}$. Quare si abscissa unitate multo maior ponatur $= P$ eique respondens applicata $= Q$, ob $p = \frac{1}{P}$ et $q = pQ = \frac{Q}{P}$ erit

$$Q = P + AP^{-1} + BP^{-3} + CP^{-5} + DP^{-7} + \text{etc.}$$
$$+ \alpha P^{-1}lP + \beta P^{-3}lP + \gamma P^{-5}lP + \delta P^{-7}lP + \text{etc.}$$

Hinc si abscissa P fiat infinita, erit

$$Q = P + \frac{\alpha lP}{P} \quad \text{seu} \quad Q - P = \frac{\alpha lP}{P},$$

unde natura rami Dq in infinitum extensi et ad asymtotam CV appropinquantis colligitur.

51. Quia porro novimus, si $p = 1$, fore $q = \frac{\pi}{2}$, pro hoc casu aequatio inventa hanc formam ob $l1 = 0$ induet

$$\frac{\pi}{2} = 1 + A + B + C + D + E + \text{etc.}$$

Cum igitur valor A nondum sit definitus, reliqui vero B, C, D etc. ab eo pendeant, haec aequatio conditionem continet, qua valor ipsius A determinatur. Ita scilicet valorem ipsius A comparatum esse oportet, ut summa seriei infinitae $1 + A + B + C +$ etc. fiat $\frac{\pi}{2}$. Verum si valores reliquarum litterarum B, C, D etc., qui ab A pendent, evolvantur, tam complicatae resultant expressiones, ut hinc valor ipsius A neutiquam erui possit.

52. Ad hanc constantem A determinandam alia patet via, si datae cuiuspiam ellipsis perimeter ex altera formula in numeris fuerit inventa. Quae methodus cum requirat, ut omnes coefficientes in fractionibus decimalibus evolvantur, computo peracto reperietur

$\alpha = 0{,}5000000000;$	A quaeritur
$\beta = 0{,}1875000000;$	$B = 0{,}3750000000\,A - 0{,}1093750000$
$\gamma = 0{,}1171875000;$	$C = 0{,}2343750000\,A - 0{,}0820312500$
$\delta = 0{,}0854492188;$	$D = 0{,}1708984375\,A - 0{,}0641886393$
$\varepsilon = 0{,}0672912598;$	$E = 0{,}1345825195\,A - 0{,}0524978638$
$\zeta = 0{,}0555152893;$	$F = 0{,}1110305786\,A - 0{,}0443481445$
$\eta = 0{,}0472540855;$	$G = 0{,}0945081711\,A - 0{,}0383663416$
$\theta = 0{,}0411363691;$	$H = 0{,}0822727382\,A - 0{,}0337966961$ [1])
$\iota = 0{,}0364228268;$	$I = 0{,}0728456536\,A - 0{,}0301949487$
$\varkappa = 0{,}0326793696;$	$K = 0{,}0653587392\,A - 0{,}0272843726$
	etc.

Hisque valoribus inventis, si abscissa sit $CP = p$, valor applicatae q ita definietur, ut sit

$$q = 1 + Ap^2 + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + Gp^{14} + Hp^{16} + Ip^{18} + Kp^{20} + \text{etc.}$$
$$- pplp(\alpha + \beta p^2 + \gamma p^4 + \delta p^6 + \varepsilon p^8 + \zeta p^{10} + \eta p^{12} + \theta p^{14} + \iota p^{16} + \varkappa p^{18} + \text{etc.}).$$

1) Editio princeps: 0,0337966962. Correxit A. K.

53. Deinde vero supra eiusdem applicatae q valorem ita invenimus expressum, ut sit

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2}(1-pp) - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}(1-pp)^2 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}(1-pp)^3 - \text{etc.}\right).$$

Nunc igitur ex utraque formula pro eodem quopiam valore ipsius p eruamus valorem ipsius q, ut deinceps ex aequalitate horum duorum elicere queamus valorem coefficientis A. Pro p vero non nimis exiguam fractionem substitui conveniet, ne expressio posterior nimis lente convergat; tam parvum tamen assumamus, ut coefficientes pro superiore forma computati valori q ad 10 figuras inveniendo sufficiant.

54. Ponamus ergo ad commodum calculi $p = \frac{1}{5}$; erit in logarithmis hyperbolicis

$$-lp = 1{,}60943791243.$$

Iam vero est

$$\begin{aligned}
\alpha pp &= 0{,}02000000000\\
\beta p^4 &= 0{,}00030000000\\
\gamma p^6 &= 0{,}00000750000\\
\delta p^8 &= 0{,}00000021875\\
\varepsilon p^{10} &= 0{,}00000000689\\
\zeta p^{12} &= 0{,}00000000023\\
\eta p^{14} &= \underline{0{,}00000000001}\\
&\ 0{,}02030772588 \quad \text{coefficiens ipsius } -lp\\
&\ \underline{1{,}60943791243}\\
&\ 0{,}03268402394 \quad \text{productum.}
\end{aligned}$$

Deinde est

$$\begin{aligned}
Ap^2 &= 0{,}04000000000\,A\\
Bp^4 &= 0{,}00060000000\,A - 0{,}00017500000\\
Cp^6 &= 0{,}00001500000\,A - 0{,}00000525000\\
Dp^8 &= 0{,}00000043750\,A - 0{,}00000016432\\
Ep^{10} &= 0{,}00000001378\,A - 0{,}00000000538\\
Fp^{12} &= 0{,}00000000045\,A - 0{,}00000000018\,{}^{1)}\\
Gp^{14} &= \underline{0{,}00000000002\,A - 0{,}00000000001}\\
&\ 0{,}04061545175\,A - 0{,}00018041989\,{}^{2)}
\end{aligned}$$

1) Editio princeps: 0,00000000016. 2) Editio princeps: 0,00018041987. Correxit A. K.

Ex his conficitur

$$q = 0{,}04061545175\,A + 1{,}03250360409\ ^{1)}.$$

55. Nunc eundem valorem ipsius q ex altera aequatione quaeramus, et cum sit $p = \frac{1}{5}$, erit $1 - pp = \frac{24}{25}$; sit $nn = \frac{24}{25}$, erit

$$q = \frac{\pi}{2}\left(1 - \frac{1\cdot 1}{2\cdot 2}nn - \frac{1\cdot 1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}n^4 - \frac{1\cdot 1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}n^6 - \text{etc.}\right);$$

ponatur ad abbreviandum

$$q = \frac{\pi}{2} - \mathfrak{A}n^2 - \mathfrak{B}n^4 - \mathfrak{C}n^6 - \mathfrak{D}n^8 - \mathfrak{E}n^{10} - \text{etc.}$$

Verum hoc casu ob $nn = \frac{24}{25}$ series ista nimis lente convergit, quam ut hinc valor ipsius q satis exacte elici queat; quare, ut utrinque parem convergentiam obtineamus, ponamus $p = \frac{1}{\sqrt{2}}$, ut sit tam $pp = \frac{1}{2}$ quam $nn = \frac{1}{2}$; calculum vero tantum ad 6 figuras expediamus eritque

$$\begin{aligned}
App &= 0{,}500000\,A \\
Bp^4 &= 0{,}093750\,A - 0{,}027344 \\
Cp^6 &= 0{,}029297\,A - 0{,}010254 \\
Dp^8 &= 0{,}010681\,A - 0{,}004012 \\
Ep^{10} &= 0{,}004206\,A - 0{,}001640 \\
Fp^{12} &= 0{,}001735\,A - 0{,}000693 \\
Gp^{14} &= 0{,}000738\,A - 0{,}000300 \\
Hp^{16} &= 0{,}000321\,A - 0{,}000132 \\
Ip^{18} &= 0{,}000142\,A - 0{,}000059 \\
Kp^{20} &= 0{,}000064\,A - 0{,}000026 \\
\text{Summa reliquorum: } & \ 60\,A - \qquad 24 \\
\hline
\text{Summa omnium } & 0{,}640994\,A - 0{,}044484 + 0{,}320497\,l\frac{1}{p} + 1
\end{aligned}$$

ergo

$$q = 1{,}066592 + 0{,}640994\,A,$$

1) Editio princeps: 1,03250360407. Correxit A. K.

at altera expressio dat $q = 1{,}350647$, unde fit

$$A = \frac{284055}{640994} = 0{,}443147.$$

56. Quanquam hic valor non ultra 6 figuras extenditur, tamen casui non tribuendum videtur, quod iste numerus inventus 0,443147 a logarithmo binarii 0,69314718 unitatis quadrante 0,25 praecise deficiat. Quae coniectura si veritati esset consentanea, valorem litterae A ad plurimas figuras exhibere liceret; cum enim sit

$$l2 = 0{,}6931471805599453094172321,$$

foret $A = l2 - \frac{1}{4}$ ideoque

$$A = 0{,}4431471805599453094172321.$$

Quod autem valor coefficientis huius A sit revera $= l2 - \frac{1}{4}$, sequenti modo demonstro hancque coniecturam confirmo.

57. Comparo scilicet arcum ellipticum AYP (Fig. 2), cuius semiaxes $AC = 1$, $CP = p$, cum arcu parabolico AZS super eodem axe AC descripto, qui in A cum ellipsi communem habeat curvaturam. Sumta abscissa communi $AX = x$ sit applicata ellipsis $XY = y$ et parabolae $XZ = z$; erit

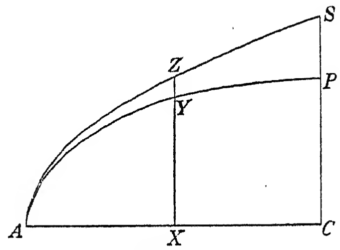

Fig. 2.

$$y = p\sqrt{(2x - xx)} \quad \text{et} \quad z = p\sqrt{2x}$$

ideoque

$$dy = \frac{p\,dx(1-x)}{\sqrt{(2x - xx)}} \quad \text{et} \quad dz = \frac{p\,dx}{\sqrt{2x}},$$

unde fit arcus ellipticus

$$AY = \int dx \sqrt{\left(1 + \frac{pp(1-x)^2}{2x - xx}\right)},$$

arcus parabolicus

$$AZ = \int dx \sqrt{\left(1 + \frac{pp}{2x}\right)}.$$

Constat autem esse

$$AZ = x\sqrt{\left(1+\frac{pp}{2x}\right)} + \frac{1}{4}ppl\frac{\sqrt{\left(1+\frac{pp}{2x}\right)}+1}{\sqrt{\left(1+\frac{pp}{2x}\right)}-1}.$$

Hinc, si ponatur $x = 1$, erit arcus parabolicus

$$AZS = \sqrt{\left(1+\frac{1}{2}pp\right)} + \frac{1}{4}ppl\frac{\sqrt{\left(1+\frac{1}{2}pp\right)}+1}{\sqrt{\left(1+\frac{1}{2}pp\right)}-1}.$$

At in formulis integralibus erit

$$\sqrt{\left(1+\frac{pp(1-x)^2}{2x-xx}\right)} = \sqrt{\left(1+\frac{pp}{2x} - \frac{pp(3-2x)}{4-2x}\right)}.$$

Quia autem comparationem non ad altiores ipsius p potestates extendere opus est quam ad secundam, coefficientes enim altiorum ipsius p potestatum ex minoribus iam definivimus, reiectis terminis, qui continent p^4 et altiores potestates, erit

$$\sqrt{\left(1+\frac{pp(1-x)^2}{2x-xx}\right)} = \sqrt{\left(1+\frac{pp}{2x}\right)} - \frac{pp(3-2x)}{4(2-x)}$$

ideoque

$$AY = \int dx\sqrt{\left(1+\frac{pp}{2x}\right)} - \frac{1}{4}pp\int\frac{dx(3-2x)}{2-x}$$

integralibusque actu sumtis

$$AY = x\sqrt{\left(1+\frac{pp}{2x}\right)} + \frac{1}{4}ppl\frac{\sqrt{\left(1+\frac{pp}{2x}\right)}+1}{\sqrt{\left(1+\frac{pp}{2x}\right)}-1} - \frac{1}{2}ppx - \frac{1}{4}ppl\frac{2-x}{2}.$$

Ponatur iam $x = 1$, ut prodeat arcus $AYP = q$; erit

$$q = \sqrt{\left(1+\frac{1}{2}pp\right)} + \frac{1}{4}ppl\left(\sqrt{\left(1+\frac{1}{2}pp\right)}+1\right)$$
$$-\frac{1}{4}ppl\left(\sqrt{\left(1+\frac{1}{2}pp\right)}-1\right) - \frac{1}{2}pp + \frac{1}{4}ppl2.$$

58. Iam quoniam ad altiores ipsius p potestates non respicimus, erit

$$\sqrt{\left(1+\frac{1}{2}pp\right)} = 1 + \frac{1}{4}pp,$$

unde fiet

$$q = 1 + \frac{1}{4}pp + \frac{1}{4}ppl\left(2 + \frac{1}{4}pp\right) - \frac{1}{4}ppl\frac{1}{4}pp - \frac{1}{2}pp + \frac{1}{4}ppl2,$$

ubi pro $l\left(2 + \frac{1}{4}pp\right) = l2 + \frac{1}{8}pp$ scribere licet $l2$, ita ut sit

$$q = 1 - \frac{1}{4}pp + \frac{1}{2}ppl2 - \frac{1}{2}pplp + \frac{1}{2}ppl2$$

seu

$$q = 1 - \frac{1}{2}pplp + pp\left(l2 - \frac{1}{4}\right),$$

unde perspicitur coefficientem ipsius pp, quem ante littera A indicavimus, esse $= l2 - \frac{1}{4}$, omnino uti ex casu ante computato coniectura sumus consecuti.

59. Pro curva igitur initio proposita $AQDq$ (Fig. 1, p. 22), si fuerit abscissa $CP = p$ et applicata $PQ = q$, erit

$$\begin{aligned} q = 1 &+ App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + \text{etc.} \\ &- (\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \text{etc.})\, lp, \end{aligned}$$

ubi coefficientes ita determinantur

$$\begin{array}{ll}
A = l2 - \frac{1}{4}; & \alpha = \frac{1}{2} \\[2mm]
B = \frac{1 \cdot 3}{2 \cdot 4}A - \frac{1}{2}(\alpha - \beta) + \frac{1}{2} \cdot \frac{\beta}{2}; & \beta = \frac{1 \cdot 3}{2 \cdot 4}\alpha \\[2mm]
C = \frac{3 \cdot 5}{4 \cdot 6}B - \frac{1}{3}(\beta - \gamma) + \frac{1}{4} \cdot \frac{\gamma}{3}; & \gamma = \frac{3 \cdot 5}{4 \cdot 6}\beta \\[2mm]
D = \frac{5 \cdot 7}{6 \cdot 8}C - \frac{1}{4}(\gamma - \delta) + \frac{1}{6} \cdot \frac{\delta}{4}; & \delta = \frac{5 \cdot 7}{6 \cdot 8}\gamma \\[2mm]
E = \frac{7 \cdot 9}{8 \cdot 10}D - \frac{1}{5}(\delta - \varepsilon) + \frac{1}{8} \cdot \frac{\varepsilon}{5}; & \varepsilon = \frac{7 \cdot 9}{8 \cdot 10}\delta \\[2mm]
F = \frac{9 \cdot 11}{10 \cdot 12}E - \frac{1}{6}(\varepsilon - \zeta) + \frac{1}{10} \cdot \frac{\zeta}{6}; & \zeta = \frac{9 \cdot 11}{10 \cdot 12}\varepsilon
\end{array}$$

etc.

Series haec valde convergit, si abscissa p fuerit fractio valde parva, sin autem sit unitate multo maior, iisdem manentibus coefficientibus erit

$$q = p + \frac{A}{p} + \frac{B}{p^3} + \frac{C}{p^5} + \frac{D}{p^7} + \frac{E}{p^9} + \text{etc.}$$
$$+ \left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\varepsilon}{p^9} + \text{etc.}\right) lp.$$

60. Verum si abscissa p non multum ab unitate discrepet, uti conveniet hac serie supra § 26 inventa

$$q = 1 + pp \left\{ \begin{array}{l} \left(\frac{\pi}{2} - 1\right) + \left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2} - 1\right)(1-pp) + \left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\cdot\frac{\pi}{2} - 1\right)(1-pp)^2 \\ \qquad + \left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot\frac{\pi}{2} - 1\right)(1-pp)^3 + \text{etc.} \end{array} \right\}$$

quae etiam ex natura ellipsis in hanc convertitur

$$q = p + \frac{1}{p} \left\{ \begin{array}{l} \left(\frac{\pi}{2} - 1\right) - \left(\frac{1\cdot 3}{2\cdot 2}\cdot\frac{\pi}{2} - 1\right)\frac{(1-pp)}{pp} + \left(\frac{1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4}\cdot\frac{\pi}{2} - 1\right)\frac{(1-pp)^2}{p^4} \\ \qquad - \left(\frac{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}\cdot\frac{\pi}{2} - 1\right)\frac{(1-pp)^3}{p^6} + \text{etc.} \end{array} \right\}$$

unde, prout fuerit vel $p > 1$ vel $p < 1$, eam eligere licet, cuius termini vel iisdem signis procedant vel alternantibus. Plerumque autem praestat ad summam proxime definiendam signa eligere alternantia.

PROBLEMA

61. *Datis axibus coniugatis ellipsis in numeris proxime exhibere eius perimetrum.*

SOLUTIO

Sint semiaxes ellipsis 1 et p et quadrans perimetri $= q$ atque per formulas inventas valor ipsius q in numeris definiri poterit, dummodo ea eligatur, cuius termini maxime convergant. Quatuor autem adepti sumus formulas, quae sunt

$$\text{I.}\quad q = 1 + App + Bp^4 + Cp^6 + Dp^8 + Ep^{10} + Fp^{12} + \text{etc.}$$
$$- (\alpha pp + \beta p^4 + \gamma p^6 + \delta p^8 + \varepsilon p^{10} + \zeta p^{12} + \text{etc.})\, lp$$

$$\text{II.}\quad q = p + A\frac{1}{p} + B\frac{1}{p^3} + C\frac{1}{p^5} + D\frac{1}{p^7} + E\frac{1}{p^9} + F\frac{1}{p^{11}} + \text{etc.}$$
$$+\left(\frac{\alpha}{p} + \frac{\beta}{p^3} + \frac{\gamma}{p^5} + \frac{\delta}{p^7} + \frac{\varepsilon}{p^9} + \frac{\zeta}{p^{11}} + \text{etc.}\right) lp$$

$$\text{III.}\quad q = 1 + pp\left(\mathfrak{A} + \mathfrak{B}(1-pp) + \mathfrak{C}(1-pp)^2 + \mathfrak{D}(1-pp)^3 + \mathfrak{E}(1-pp)^4 + \text{etc.}\right)$$

$$\text{IV.}\quad q = p + \frac{1}{p}\left(\mathfrak{A} - \mathfrak{B}\frac{(1-pp)}{pp} + \mathfrak{C}\frac{(1-pp)^2}{p^4} - \mathfrak{D}\frac{(1-pp)^3}{p^6} + \mathfrak{E}\frac{(1-pp)^4}{p^8} - \text{etc.}\right).$$

Horum autem tergeminorum coefficientium valores sunt in numeris

$A = 0{,}44314718056$	$\alpha = 0{,}50000000000$	$\mathfrak{A} = 0{,}57079632679$
$B = 0{,}05680519271$	$\beta = 0{,}18750000000$	$\mathfrak{B} = 0{,}17809724510$
$C = 0{,}02183137044$	$\gamma = 0{,}11718750000$	$\mathfrak{C} = 0{,}10446616728$
$D = 0{,}01154452144$ [1]	$\delta = 0{,}08544921875$	$\mathfrak{D} = 0{,}07378655152$
$E = 0{,}00714200029$	$\varepsilon = 0{,}06729125977$	$\mathfrak{E} = 0{,}05700863665$
$F = 0{,}00485474337$	$\zeta = 0{,}05551528931$ [2]	$\mathfrak{F} = 0{,}04643855029$
$G = 0{,}00351468795$	$\eta = 0{,}04725408554$	$\mathfrak{G} = 0{,}03917161591$
$H = 0{,}00266223578$	$\theta = 0{,}04113636911$	$\mathfrak{H} = 0{,}03386971991$
$I = 0{,}00208639732$	$\iota = 0{,}03642282682$	$\mathfrak{I} = 0{,}02983116632$
$K = 0{,}00167916842$	$\varkappa = 0{,}03267936962$	$\mathfrak{K} = 0{,}02665267507$
		$\mathfrak{L} = 0{,}02408604338$ [3]

Hinc pro quavis ellipsis specie habebitur series convergens, unde eius perimeter definiri poterit; veluti si ponatur

$$p = \frac{1}{10}, \quad \text{erit} \quad q = 1{,}015993545021,$$

si sit

$$p = \frac{1}{5}, \quad \text{erit} \quad q = 1{,}05050222700,$$

si sit

$$p = \frac{1}{\sqrt{2}}, \quad \text{erit} \quad q = 1{,}3506429.$$

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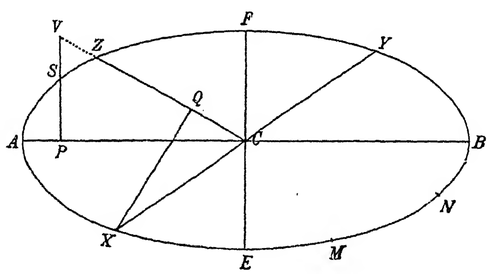
1) Editio princeps: 0,01154452143. 2) Editio princeps: 0,05551527931. 3) Editio princeps: 0,02408604339. Correxit A. K.

PROBLEMA AD CUIUS SOLUTIONEM GEOMETRAE INVITANTUR
THEOREMA AD CUIUS DEMONSTRATIONEM GEOMETRAE INVITANTUR

Commentatio 211 indicis ENESTROEMIANI
Nova acta eruditorum 1754, p. 40

PROBLEMA
AD CUIUS SOLUTIONEM GEOMETRAE INVITANTUR

Proposito quadrante elliptico $BNME$ inter binos semiaxes principales CB et CE intercepto in eo geometrice assignare puncta M et N, ut arcus MN praecise sit semissis arcus quadrantis $BNME$.



THEOREMA
AD CUIUS DEMONSTRATIONEM GEOMETRAE INVITANTUR

Si ellipsis $AEBF$ axibus principalibus AB et EF descripta per diametrum quamcumque obliquam XCY bisecetur, ad quam semidiameter coniugata CZ producatur in V, ut sit CV aequalis semiaxi CA, et ex V ad CA normalis agatur

VP, haec ellipsin ita secabit in puncto S, ut arcuum XAS et YFS differentia geometrice assignari possit.

Si enim ex X ad CZ perpendiculum XQ ducatur, intervallum CQ bis sumtum aequale erit illorum arcuum differentiae seu erit

$$YFS - XAS = 2CQ.$$

Eo difficilius autem tam Problema resolvendum videtur quam Theorema demonstrandum, quod diversi arcus elliptici nullo adhuc modo inter se comparari potuerint, unde ex harum propositionum pertractatione non contemnenda Analyseos incrementa merito exspectantur. Graviore autem praemio Geometrae ad hoc argumentum suscipiendum incitari non possunt.

Solutio problematis et demonstratio theorematis inveniuntur in L. EULERI Commentatione 264 (indicis ENESTROEMIANI); vide p. 201. A. K.

PROBLEMA AD CUIUS SOLUTIONEM GEOMETRAE INVITANTUR
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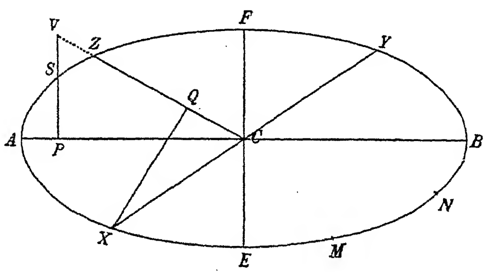

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Solutio problematis et demonstratio theorematis inveniuntur in L. EULERI Commentatione 264 (indicis ENESTROEMIANI); vide p. 201. A. K.

DE INTEGRATIONE AEQUATIONIS DIFFERENTIALIS

$$\frac{m\,dx}{\sqrt{(1-x^4)}}=\frac{n\,dy}{\sqrt{(1-y^4)}}$$

Commentatio 251 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1761, p. 37—57
Summarium ibidem p. 7—9

SUMMARIUM

In hac dissertatione et nonnullis sequentibus, quibus simile argumentum pertractatur, quasi novus plane campus in Analysi aperitur integralia diversarum formularum, quae per se omnem integrationis solertiam respuunt, inter se comparandi. Cum enim ope notae comparationis angulorum relatio inter binas variabiles x et y huic aequationi differentiali

$$\frac{m\,dx}{\sqrt{(1-xx)}}=\frac{n\,dy}{\sqrt{(1-yy)}}$$

conveniens algebraice exhiberi queat, etsi utraque formula per se algebraice integrari nequit, sed angulum seu arcum circularem exprimit, haec relatio ex eo tantum fonte petita videtur, quod angulorum datam et quidem rationalem rationem tenentium sinus algebraice inter se comparari possunt. Neque talis comparatio locum habere videtur, nisi ambae formulae sive per angulos sive per logarithmos integrari queant. Quoties quidem solutio cuiusquam problematis ad huiusmodi aequationem differentialem $Xdx=Ydy$, in qua X sit functio ipsius x et Y ipsius y, tantum perducitur, ea, quia variabiles sunt a se invicem separatae, tanquam penitus absoluta spectari solet, cum ope quadraturae duarum curvarum, quarum alterius area per $\int Xdx$, alterius per $\int Ydy$ exprimitur, construi posset. Verum si pro dato quovis valore ipsius x valor ipsius y conveniens assignari debeat, id utramque quadraturam involvere videtur, sine qua relatio inter x et y minime exhiberi queat. Multo magis igitur mirum videbitur, cum talis formulae $\frac{dz}{\sqrt{(1-z^4)}}$ integrale neque per angulos neque per logarithmos exprimi possit, quae quantitates transcendentes ad comparationem solae idoneae putantur, nihilominus pro aequatione differentiali proposita relationem inter x

et y algebraice exhiberi posse, ita ut linea curva, cuius arcus indefinite hac formula integrali $\int \frac{dz}{\sqrt{(1-z^4)}}$ exprimitur, pari proprietate ac circulus sit praedita, ut scilicet omnes eius arcus inter se comparari seu proposito in eo arcu quocunque alius arcus, qui ad eam datam teneat rationem, geometrice assignari queat. Vel, quod eodem redit, aequatio integralis aequationis differentialis propositae, quae veram relationem inter x et y exprimit, non solum non tale integrale involvet, sed adeo erit algebraica.

Atque hoc quidem non tantum pro casu quodam particulari, verum adeo integrale completum, quod quantitatem constantem arbitrariam complectitur, erit algebraicum. Neque vero talis admiranda integratio in ipsa tantum aequatione differentiali locum habet, sed simili omnino modo Cel. Auctor ostendit hanc aequationem differentialem multo latius patentem

$$\frac{m\,dx}{\sqrt{(A+Bx^2+Cx^4)}} = \frac{n\,dy}{\sqrt{(A+By^2+Cy^4)}}$$

per aequationem algebraicam complete integrari posse, si modo numeri m et n sint rationales; quin etiam eandem integrandi methodum ad hanc aequationem multo generaliorem extendit

$$\frac{m\,dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}} = \frac{n\,dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}},$$

ubi in denominatoribus radicalibus omnes potestates ipsarum x et y ad quartam usque occurrunt. Hinc suspicari liceret, etiamsi hae potestates altius ascenderent, integrationem tamen algebraicam adhuc locum esse habituram; sed praeterquam quod methodus Auctoris in ipsa potestate quarta terminatur, facile ostendi potest, in potestate certe sexta algebraicam integrationem in genere excludi. Si enim coefficientes ita accipiantur, ut radix quadrata extrahi queat, ex hoc solo casu $\frac{m\,dx}{1+x^3} = \frac{n\,dy}{1+y^3}$ evidens est relationem inter x et y nequaquam algebraice exprimi posse, cum utriusque formulae integrale tam angulum quam logarithmum involvat; anguli autem et logarithmi certe inter se algebraice comparari non patiuntur. Interim tamen peculiari modo integratio huius quoque aequationis

$$\frac{m\,dx}{\sqrt{(A+Bx^2+Cx^4+Dx^6)}} = \frac{n\,dy}{\sqrt{(A+By^2+Cy^4+Dy^6)}}$$

algebraice exhibetur, unde patet hanc dissertationem multo plures investigationes continere, quam titulus quidem prae se ferre videtur.

1. Cum primum occasione inventionum Ill. Comitis FAGNANI[1]) hanc aequationem essem contemplatus, eiusmodi quidem relationem algebraicam inter

1) G. C. FAGNANO (1682—1766), *Produzioni matematiche*, T. 2, Pesaro 1750; *Opere matematiche*, T. 2, Milano-Roma-Napoli 1911. A. K.

variabiles x et y elicui, quae huic aequationi satisfaceret; sed ea relatio non pro aequatione integrali completa haberi poterat, propterea quod non complecteretur quantitatem constantem arbitrariam, cuiusmodi semper in calculum per integrationem introduci solet. Hinc enim, uti satis notum est, integralia incompleta et particularia distingui solent, quorum illa totam vim aequationum differentialium exhauriunt, haec vero tantum ita satisfaciunt, ut aliae insuper expressiones aeque satisfacere queant. Criterium autem aequationis integralis completae in hoc consistit, quod ea quantitatem constantem involvere debeat, quae in aequatione differentiali non apparet.

2. Quae quo clarius perspiciantur, sufficiet aequationem differentialem simplicissimam $dx = dy$ considerasse, cui utique satisfacit haec integralis $x = y$; in rem tamen haec integralis minus late patet quam differentialis $dx = dy$, cum huic aeque satisfaciat haec integralis $x = y \mp a$ multo latius patens, sumendo pro a quantitatem constantem quamcunque, atque haec demum integralis totam vim aequationis differentialis $dx = dy$ exhaurire censetur, ex quo etiam aequatio integralis completa appellatur, propterea quod in ea inest quantitas constans a, quae in aequatione differentiali non occurrit. Quodsi vero loco istius constantis indefinitae a valores determinati substituantur, ex integrali completo obtinentur integralia particularia, quae ob hanc ipsam rationem minus late patent, quam aequatio differentialis proposita.

3. Saepe numero autem aequationis differentialis integrale particulare algebraicum exhiberi potest, cum tamen integrale completum sit transcendens; hoc scilicet evenit, si pars transcendens per constantem illam arbitrariam fuerit multiplicata, quae propterea constante illa nihilo aequali posita ex calculo evanescit et integrale algebraicum particulare relinquit. Ita huic aequationi $dy = dx + (y - x)dx$ manifestum est satisfacere valorem $y = x$, quo tamen tantum integrale particulare continetur, cum completum sit $y = x + ae^x$ denotante e numerum, cuius logarithmus est $= 1$. Nisi igitur constans arbitraria a evanescens ponatur, integrale semper erit transcendens.

4. Cum igitur evenire queat, ut aequatio differentialis integrale particulare algebraicum admittat, etiamsi integrale completum sit transcendens, ita etiam rationes dubitandi non desunt, quod integrale completum aequationis differentialis propositae

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}}$$

quantitates transcendentes involvat, etiamsi pro ea integrale particulare algebraicum exhibere licuerit. Cum enim integrale completum sit

$$m\int\frac{dx}{\sqrt{(1-x^4)}} = n\int\frac{dy}{\sqrt{(1-y^4)}} + C,$$

haec autem integralia nullo modo, neque circuli neque hyperbolae quadraturam in subsidium vocando, assignari queant, minime probabile videtur istas formulas tantopere transcendentes in genere, ita ut constans C maneat indeterminata, ad relationem algebraicam inter x et y revocari posse.

5. Notum quidem est integrale completum huius aequationis differentialis

$$\frac{m\,dx}{\sqrt{(1-xx)}} = \frac{n\,dy}{\sqrt{(1-yy)}}$$

semper algebraice exhiberi posse, dummodo proportio coefficientium m et n fuerit rationalis; sed quia utriusque formulae integrale arcum circuli indicat, ita ut integrale completum sit $m\,\mathrm{A}\sin.x = n\,\mathrm{A}\sin.y + C$, relatio autem sinuum, qui ad arcus proportionem rationalem inter se tenentes spectant, algebraice exprimi potest, mirum non est aequationem integralem completam his casibus quoque algebraice exhiberi posse. Cum autem huiusmodi comparatio in formulis transcendentibus $\int\frac{dx}{\sqrt{(1-x^4)}}$ et $\int\frac{dy}{\sqrt{(1-y^4)}}$ locum non habeat seu saltem non constet, inde reductio integralis ad quantitates algebraicas peti non poterit.

6. Nihilo tamen minus observavi, si proposita fuerit huiusmodi aequatio differentialis

$$\frac{m\,dx}{\sqrt{(1-x^4)}} = \frac{n\,dy}{\sqrt{(1-y^4)}},$$

etiam integrale completum, quod scilicet quantitatem constantem arbitrariam involvat, semper algebraice exprimi posse, dummodo ratio $m:n$ fuerit rationalis; quod mihi quidem eo magis notatu dignum videtur, quod nulla certa methodo ad hoc integrale sum perductus, sed id potius tentando vel divinando elicui. Unde nullum est dubium, quin methodus directa ad idem hoc integrale perducens fines Analyseos non mediocriter sit amplificatura; cuius propterea investigatio Analystis omni studio commendanda videtur.

7. Completum autem integrale aequationis istius differentialis, quaecunque fuerit ratio rationalis coefficientium m et n, derivare mihi licuit ex integratione completa huius aequationis

$$\frac{dx}{\sqrt{(1-x^4)}}=\frac{dy}{\sqrt{(1-y^4)}};$$

hac enim concessa methodum certam indicabo ex ea quoque integrale completum huius aequationis multo latius patentis

$$\frac{m\,dx}{\sqrt{(1-x^4)}}=\frac{n\,dy}{\sqrt{(1-y^4)}}$$

concludendi. Quae methodus etiam in genere ad huiusmodi aequationum $mXdx=nYdy$ integralia invenienda adhiberi queat, si modo integrale completum huius $Xdx=Ydy$ fuerit erutum atque Y talem significet functionem ipsius y, qualis X est ipsius x.

8. Exordiar igitur ab hac aequatione

$$\frac{dx}{\sqrt{(1-x^4)}}=\frac{dy}{\sqrt{(1-y^4)}},$$

cui quidem primo intuitu satisfacere perspicuum est aequationem $x=y$, quae propterea eius est integrale particulare. Tum vero eidem aequationi quoque satisfacit iste valor algebraicus

$$x=-\sqrt{\frac{1-yy}{1+yy}};$$

cum enim sit

$$dx=+\frac{2ydy}{(1+yy)\sqrt{(1-yy)(1+yy)}}\quad\text{et}\quad\sqrt{(1-x^4)}=\frac{2y}{1+yy},$$

erit

$$\frac{dx}{\sqrt{(1-x^4)}}=\frac{dy}{\sqrt{(1-y^4)}}.$$

Hinc iste etiam valor seu aequatio $xxyy+xx+yy-1=0$ est integralis particularis aequationis differentialis propositae. Unde integrale completum, quod constantem arbitrariam involvat, ita comparatum sit necesse est, ut tribuendo huic constanti certum quendam valorem prodeat

$$x=y,$$

sin autem eidem constanti alius quidem valor tribuatur, ut prodeat

$$x=-\sqrt{\frac{1-yy}{1+yy}}\quad\text{seu}\quad xxyy+xx+yy-1=0.$$

THEOREMA

9. *Dico igitur huius aequationis differentialis*

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}$$

aequationem integralem completam esse

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)}.$$

DEMONSTRATIO

Posita enim hac aequatione eius differentiale erit

$$xdx + ydy + ccxy(xdy + ydx) = (xdy + ydx)\sqrt{(1-c^4)},$$

unde fit

$$dx(x + ccxyy - y\sqrt{(1-c^4)}) + dy(y + ccxxy - x\sqrt{(1-c^4)}) = 0.$$

Ex eadem vero aequatione resoluta colligitur

$$y = \frac{x\sqrt{(1-c^4)} + c\sqrt{(1-x^4)}}{1+ccxx} \quad \text{et} \quad x = \frac{y\sqrt{(1-c^4)} - c\sqrt{(1-y^4)}}{1+ccyy}.$$

Si enim ibi radicali $\sqrt{(1-x^4)}$ tribuitur signum $+$, hic radicali $\sqrt{(1-y^4)}$ signum $-$ tribui debet, ut posito $x = 0$ utrinque idem valor prodeat $y = c$. Erit ergo

$$x + ccxyy - y\sqrt{(1-c^4)} = -c\sqrt{(1-y^4)},$$

$$y + ccxxy - x\sqrt{(1-c^4)} = \quad c\sqrt{(1-x^4)},$$

quibus valoribus in aequatione differentiali substitutis prodit

$$-cdx\sqrt{(1-y^4)} + cdy\sqrt{(1-x^4)} = 0$$

sive

$$\frac{dx}{\sqrt{(1-x^4)}} = \frac{dy}{\sqrt{(1-y^4)}}.$$

Huius ergo aequationis differentialis integrale est

$$xx + yy + ccxxyy = cc + 2xy\sqrt{(1-c^4)},$$

et quia constantem c ab arbitrio nostro pendentem continet, erit simul integrale completum. Q. E. D.

10. Si igitur habeatur haec aequatio $\frac{dx}{\sqrt{(1-x^4)}}=\frac{dy}{\sqrt{(1-y^4)}}$, valor integralis completus ipsius x est

$$x=\frac{y\sqrt{(1-c^4)}\pm c\sqrt{(1-y^4)}}{1+ccyy},$$

unde, si constans arbitraria c evanescat, fit $x=y$; sin autem ponatur $c=1$, habemus $x=\pm\frac{\sqrt{(1-y^4)}}{1+yy}=\sqrt{\frac{1-yy}{1+yy}}$, qui sunt ambo illi valores particulares iam supra exhibiti. Hinc eruuntur alii valores particulares prae caeteris simpliciores, sed qui ad imaginaria devolvuntur. Ita posito $c=\infty$ fit

$$x=\frac{\sqrt{-1}}{y}$$

et posito $cc=-1$ fit

$$x=\sqrt{\frac{yy+1}{yy-1}},$$

qui itidem aequationi propositae satisfaciunt.

11. Quo autem ratio huius integralis clarius perspiciatur, concipiatur curva AM (Fig. 1), cuius haec sit indoles, ut posita abscissa $AP=u$ sit arcus

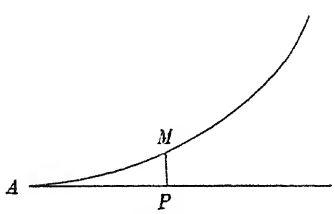

Fig. 1.

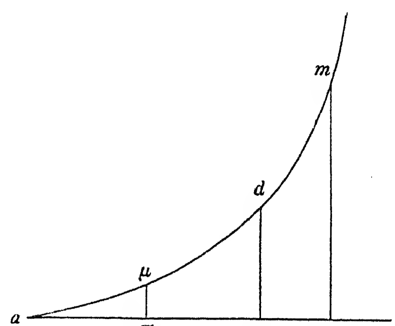

Fig. 2.

ei respondens $AM=\int\frac{du}{\sqrt{(1-u^4)}}$. Deinde eadem curva denuo (Fig. 2) descripta capiatur abscissa $ap=x$; erit arcus $am=\int\frac{dx}{\sqrt{(1-x^4)}}$. Sumto igitur

$$x=\frac{u\sqrt{(1-c^4)}\pm c\sqrt{(1-u^4)}}{1+ccuu}$$

fiet $\frac{dx}{\sqrt{(1-x^4)}}=\frac{du}{\sqrt{(1-u^4)}}$ ideoque arc. $am=$ arc. $AM+$ Const. Pro constantis

autem huius determinatione posito $u = 0$, quo casu arcus AM evanescit, fit $x = c$. Quare si capiatur abscissa $ab = c$, cui arcus ad respondeat, erit arcus $dm =$ arcui AM.

12. Ope huius ergo integrationis completae aequationis $\frac{dx}{\sqrt{(1-x^4)}} = \frac{du}{\sqrt{(1-u^4)}}$ in curva proposita arcui cuicunque AM, qui abscissae $AP = u$ respondet, arcus aequalis dm, qui a dato puncto d incipiat, abscindi poterit. Posita enim abscissa dato puncto d respondente $ab = c$ si capiatur abscissa

$$ap = x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu},$$

erit arcus dm arcui AM aequalis. Simili autem modo cum $\sqrt{(1-c^4)}$ negativum statui liceat, si capiatur abscissa

$$a\pi = \frac{c\sqrt{(1-u^4)} - u\sqrt{(1-c^4)}}{1+ccuu},$$

erit itidem arcus $d\mu$ arcui AM aequalis sicque in hac curva a dato quovis puncto d utrinque abscindi potest arcus dm et $d\mu$, qui arcui AM sint aequales.

13. Hinc ergo patet, si arcus ad aequalis capiatur arcui AM seu $c = u$, fore arcum am duplum arcus AM. Hinc si statuatur $ap = x = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$, prodibit arcus $am = 2$ arc. AM. Simili modo si capiatur arcus $ad = 2AM$ seu $c = \frac{2u\sqrt{(1-u^4)}}{1+u^4}$ statuaturque $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu}$, obtinebitur arcus $am = 3$ arc. AM. Ac si iste valor ipsius x denuo pro c substituatur, ut sit $ad = 3AM$, iterumque statuatur $x = \frac{c\sqrt{(1-u^4)} + u\sqrt{(1-c^4)}}{1+ccuu}$, nascetur arcus am quadruplus arcus AM; atque ita porro successive quaecunque multipla arcus AM geometrice assignari poterunt.

14. Sit arcus $ad = n \cdot AM$ et $ab = z$, ita ut sit

$$\int \frac{dz}{\sqrt{(1-z^4)}} = n \int \frac{du}{\sqrt{(1-u^4)}};$$

atque ex his patet, si capiatur

$$x=\frac{z\sqrt{(1-u^4)}+u\sqrt{(1-z^4)}}{1+uuzz},$$

fore

$$\int\frac{dx}{\sqrt{(1-x^4)}}=(n+1)\int\frac{du}{\sqrt{(1-u^4)}};$$

sin autem ponatur

$$x=\frac{z\sqrt{(1-u^4)}-u\sqrt{(1-z^4)}}{1+uuzz},$$

tum futurum esse

$$\int\frac{dx}{\sqrt{(1-x^4)}}=(n-1)\int\frac{du}{\sqrt{(1-u^4)}}.$$

Si igitur haec aequatio $\frac{dz}{\sqrt{(1-z^4)}}=\frac{n\,du}{\sqrt{(1-u^4)}}$ fuerit integrata debitusque valor pro z inde erutus, etiam integrari poterit haec aequatio $\frac{dx}{\sqrt{(1-x^4)}}=\frac{(n\pm 1)\,du}{\sqrt{(1-u^4)}}$, quippe cuius integrale erit $x=\frac{z\sqrt{(1-u^4)}\pm u\sqrt{(1-z^4)}}{1+uuzz}$. Ac si pro z assumtus fuerit eius valor completus, qui scilicet constantem arbitrariam involvat, etiam pro x prodibit eius valor completus.

15. Hinc igitur perspicuum est, quomodo aequatio integralis completa inveniri debeat, quae conveniat huic aequationi differentiali $\frac{dx}{\sqrt{(1-x^4)}}=\frac{n\,du}{\sqrt{(1-u^4)}}$, quoties n fuerit numerus integer. Simili autem modo assignari poterit y, ut sit $\frac{dy}{\sqrt{(1-y^4)}}=\frac{m\,du}{\sqrt{(1-u^4)}}$; unde, si eliminando u aequatio inter x et y quaeratur, ea erit integralis huius aequationis $\frac{m\,dx}{\sqrt{(1-x^4)}}=\frac{n\,dy}{\sqrt{(1-y^4)}}$, quicunque numeri rationales pro m et n substituantur; atque ut hoc integrale prodeat completum, sufficit pro altera tantum variabilium x et y valorem completum per u determinasse, cum hinc iam nova constans arbitraria in calculum introducatur.

16. Methodus, qua hic in theorematis demonstratione sum usus, etsi non ex rei natura est petita, sed indirecte ad id, quod propositum erat, perduxit, tamen multo latius patet; simili enim modo colligitur huius aequationis differentialis

$$\frac{dx}{\sqrt{(1+mxx+nx^4)}}=\frac{dy}{\sqrt{(1+myy+ny^4)}}$$

integrale completum esse

$$0 = cc - xx - yy + nccxxyy + 2xy\sqrt{(1 + mcc + nc^4)}.$$

Unde idem quod ante ratiocinium adhibendo integrale quoque completum obtinebitur huius aequationis

$$\frac{\mu dx}{\sqrt{(1+mxx+nx^4)}} = \frac{\nu dy}{\sqrt{(1+myy+ny^4)}},$$

siquidem litteris μ et ν numeri integri designentur.

17. Investigatio autem huius integrationis ita se habet: Fingatur primo pro arbitrio relatio inter variabiles x et y hac aequatione contenta

$$(1) \quad \alpha xx + \alpha yy = 2\beta xy + \gamma xxyy + \delta,$$

quae differentiata dat

$$\alpha xdx + \alpha ydy = \beta xdy + \beta ydx + \gamma xyydx + \gamma xxydy,$$

unde conficitur

$$(2) \quad dx(\alpha x - \beta y - \gamma xyy) + dy(\alpha y - \beta x - \gamma xxy) = 0.$$

Deinde ex aequatione (1) eliciantur valores utriusque variabilis

$$x = \frac{\beta y + \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)}}{\alpha - \gamma yy},$$

$$y = \frac{\beta x - \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}}{\alpha - \gamma xx}.$$

Atque hinc obtinemus

$$(3) \quad \alpha x - \beta y - \gamma xyy = \sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)},$$

$$(4) \quad \alpha y - \beta x - \gamma xxy = -\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)},$$

qui valores in aequatione (2) substituti praebebunt

$$(5) \quad \frac{dx}{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)xx + \alpha\gamma x^4)}} = \frac{dy}{\sqrt{(\alpha\delta + (\beta\beta - \alpha\alpha - \gamma\delta)yy + \alpha\gamma y^4)}},$$

cuius ergo aequationis integrale est aequatio (1).

18. Quo istas formas simpliciores reddamus, ponamus

$$\alpha\delta = A, \quad \beta\beta - \alpha\alpha - \gamma\delta = C, \quad \alpha\gamma = E$$

eritque

$$\delta = \frac{A}{\alpha}, \quad \gamma = \frac{E}{\alpha} \quad \text{et} \quad \beta = \sqrt{\left(C + \alpha\alpha + \frac{AE}{\alpha\alpha}\right)}.$$

Quare huius aequationis differentialis

$$(6) \qquad \frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}}$$

aequatio integralis est haec

$$(7) \quad \alpha(xx + yy) = \frac{A}{\alpha} + \frac{E}{\alpha}xxyy + 2xy\sqrt{\left(C + \alpha\alpha + \frac{AE}{\alpha\alpha}\right)},$$

quae simul est integralis completa.

19. Vel ponamus

$$A = f\alpha\alpha, \quad C = g\alpha\alpha \quad \text{et} \quad E = h\alpha\alpha,$$

ut habeamus hanc aequationem differentialem

$$\frac{dx}{\sqrt{(f + gxx + hx^4)}} = \frac{dy}{\sqrt{(f + gyy + hy^4)}},$$

cuius propterea aequatio integralis completa erit

$$xx + yy = f + hxxyy + 2xy\sqrt{(1 + g + fh)};$$

quae etsi novam constantem involvere non videtur, tamen est completa, cum in differentiali tantum ratio quantitatum f, g et h spectetur, ita ut pro f, g et h scribere liceat fcc, gcc et hcc, unde aequatio integralis manifesto completa prodit

$$xx + yy = fcc + hccxxyy + 2xy\sqrt{(1 + gcc + fhc^4)}$$

vel

$$f(xx + yy) = fee + heexxyy + 2xy\sqrt{f(f + gee + he^4)}$$

posito $cc = \frac{ee}{f}$.

20. Quodsi ergo proposita sit haec aequatio differentialis

$$\frac{dx}{\sqrt{(f+gxx+hx^4)}} = \frac{dy}{\sqrt{(f+gyy+hy^4)}},$$

valor ipsius y per functionem algebraicam ipsius x exprimi poterit, ita ut sit

$$y = \frac{x\sqrt{(1+gcc+fhc^4)} \pm c\sqrt{(1+gxx+fhx^4)}}{1-hccxx}$$

vel

$$y = \frac{x\sqrt{f(f+gee+he^4)} \pm e\sqrt{f(f+gxx+hx^4)}}{f-heexx}.$$

Quodsi ergo fit $g=0$, ut habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{(f+hx^4)}} = \frac{dy}{\sqrt{(f+hy^4)}},$$

valor integralis completus ipsius y erit

$$y = \frac{x\sqrt{f(f+he^4)} \pm e\sqrt{f(f+hx^4)}}{f-heexx},$$

unde constantem e pro lubitu determinando innumeri valores particulares pro y deduci possunt.

21. Methodi autem, qua supra usus sum, beneficio etiam huius aequationis

$$\frac{mdx}{\sqrt{(f+gxx+hx^4)}} = \frac{ndy}{\sqrt{(f+gyy+hy^4)}},$$

si modo m et n sint numeri rationales, integrale completum atque id quidem algebraice exhiberi poterit.

22. Quemadmodum in aequatione supra assumta variabiles x et y inter se permutabiles sunt constitutae, ut ambae formulae inter se similes evaderent, ita omissa hac limitatione ad formularum differentialium disparium comparationem perveniemus. Ponamus ergo

$$(1) \quad \alpha xx + \beta yy = 2\gamma xy + \delta xxyy + \varepsilon,$$

unde fit

$$x = \frac{\gamma y + \sqrt{(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4)}}{\alpha - \delta yy}$$

et

$$y = \frac{\gamma x - \sqrt{(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4)}}{\beta - \delta xx}$$

hincque

$$(2) \quad \alpha x - \gamma y - \delta xyy = \sqrt{(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)\, yy + \beta\delta y^4)},$$

$$(3) \quad \beta y - \gamma x - \delta xxy = -\sqrt{(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)\, xx + \alpha\delta x^4)};$$

at aequatio (1) differentiata dat

$$dx(\alpha x - \gamma y - \delta xyy) + dy(\beta y - \gamma x - \delta xxy) = 0,$$

unde conficitur haec aequatio differentialis

$$\frac{dx}{\sqrt{(\beta\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)xx + \alpha\delta x^4)}} = \frac{dy}{\sqrt{(\alpha\varepsilon + (\gamma\gamma - \delta\varepsilon - \alpha\beta)yy + \beta\delta y^4)}},$$

cuius propterea integralis est aequatio assumta.

23. Verum haec disparitas facile tollitur loco y ponendo $z\sqrt{\frac{\alpha}{\beta}}$, cuius rei ratio statim ex aequatione assumta potuisset esse manifesta. Sed alia patet via ad formulas dispares perveniendi, cuius hic exemplum tradidisse sufficiat. Assumatur aequatio

$$x^4 + 2axxyy + 2bxx = c,$$

cuius differentiale est

$$dx(x^3 + axyy + bx) + axxydy = 0$$

seu

$$\frac{dx}{xy} = \frac{-a\,dy}{xx + ayy + b}.$$

Iam ex aequatione assumta primo determinetur xy per x sicque fiet

$$xy = \sqrt{\frac{c - 2bxx - x^4}{2a}},$$

tum vero $xx + ayy + b$ per y; at ob $(xx + ayy + b)^2 = c + (ayy + b)^2$ erit

$$xx + ayy + b = \sqrt{(c + (ayy + b)^2)}.$$

Quocirca habebitur aequatio differentialis ista

$$\frac{dx\sqrt{2a}}{\sqrt{(c-2bxx-x^4)}}=\frac{-a\,dy}{\sqrt{(c+bb+2abyy+aay^4)}},$$

cuius propterea integralis est assumta seu $y=\frac{\sqrt{(c-2bxx-x^4)}}{x\sqrt{2a}}$.

24. Etsi hoc integrale non est completum, tamen ex superioribus facile completum reddetur. Ponatur enim

$$\frac{a\,dy}{\sqrt{(c+bb+2abyy+aay^4)}}=\frac{a\,dz}{\sqrt{(c+bb+2abzz+aaz^4)}};$$

ob $f=c+bb,\ g=2ab,\ h=aa$ erit

$$y=\frac{z\sqrt{(c+bb)(c+bb+2abee+aae^4)}\pm e\sqrt{(c+bb)(c+bb+2abzz+aaz^4)}}{c+bb-aaeezz};$$

hic ergo valor aequalis statuatur ipsi $\frac{\sqrt{(c-2bxx-x^4)}}{x\sqrt{2a}}$ et aequatio hinc inter x et z resultans integralis erit completa huius aequationis differentialis

$$\frac{dx\sqrt{2a}}{\sqrt{(c-2bxx-x^4)}}=\frac{-a\,dz}{\sqrt{(c+bb+2abzz+aaz^4)}}.$$

Quin etiam ex allatis patet, si haec bina membra insuper per numeros rationales quoscunque multiplicentur, quemadmodum integrale completum inveniri oporteat.

25. Verum missa membrorum disparitate formationem parium membrorum generalius concipiamus; ponatur ergo

$$(1)\quad 0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy,$$

unde differentiando obtinetur

$$dx(\beta+\gamma x+\delta y+2\varepsilon xy+\varepsilon yy+\zeta xyy)+dy(\beta+\gamma y+\delta x+2\varepsilon xy+\varepsilon xx+\zeta xxy)=0$$

ideoque

$$(2)\quad \frac{dy}{\beta+\gamma x+\delta y+2\varepsilon xy+\varepsilon yy+\zeta xyy}=\frac{-dx}{\beta+\gamma y+\delta x+2\varepsilon xy+\varepsilon xx+\zeta xxy}.$$

Ex resolutione autem aequationis assumtae elicitur

$$y=\frac{-\beta-\delta x-\varepsilon xx\pm\sqrt{(\beta\beta-\alpha\gamma+2(\beta\delta-\alpha\varepsilon-\beta\gamma)x+(\delta\delta-\gamma\gamma-\alpha\zeta-2\beta\varepsilon)xx+2(\delta\varepsilon-\beta\zeta-\gamma\varepsilon)x^3+(\varepsilon\varepsilon-\gamma\zeta)x^4)}}{\gamma+2\varepsilon x+\zeta xx}.$$

Ponatur brevitatis gratia

$$\begin{array}{lll}\beta\beta-\alpha\gamma=A, & \beta\delta-\alpha\varepsilon-\beta\gamma=B, & \\ & & \delta\delta-\gamma\gamma-\alpha\zeta-2\beta\varepsilon=C \\ \varepsilon\varepsilon-\gamma\zeta=E, & \delta\varepsilon-\beta\zeta-\gamma\varepsilon=D, & \end{array}$$

eritque

$$\beta+\delta x+\varepsilon xx+\gamma y+2\varepsilon xy+\zeta xxy=\pm\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)},$$
$$\beta+\delta y+\varepsilon yy+\gamma x+2\varepsilon xy+\zeta xyy=\mp\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}.$$

26. Hinc itaque concludimus huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

aequationem integralem eamque completam esse

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy,$$

adhibita scilicet superiori horum coefficientium determinatione. Primum autem definiatur β vel ε ex hac aequatione

$$\frac{BB(\varepsilon\varepsilon-E)-DD(\beta\beta-A)}{A\varepsilon\varepsilon-E\beta\beta}+\frac{2AD\varepsilon-2BE\beta}{B\varepsilon-D\beta}=C;$$

tum vero erit

$$\gamma=\frac{A\varepsilon\varepsilon-E\beta\beta}{B\varepsilon-D\beta},\quad \alpha=\frac{\beta\beta-A}{\gamma},\quad \zeta=\frac{\varepsilon\varepsilon-E}{\gamma}$$

et

$$\delta=\frac{B\beta(\varepsilon\varepsilon-E)-D\varepsilon(\beta\beta-A)}{A\varepsilon\varepsilon-E\beta\beta}+\gamma\quad\text{seu}\quad\delta=\gamma+\frac{B+\alpha\varepsilon}{\beta}.$$

27. Hinc ergo perspicuum est etiam hanc aequationem differentialem

$$\frac{dx}{\sqrt{(A+2Dx^3)}}=\frac{dy}{\sqrt{(A+2Dy^3)}}$$

integrari posse; nam ob $B=0$, $C=0$ et $E=0$ erit

$$\frac{-DD(\beta\beta-A)}{A\varepsilon\varepsilon}-\frac{2A\varepsilon}{\beta}=0 \quad \text{seu} \quad \varepsilon=\sqrt[3]{\frac{DD}{2AA}\beta(A-\beta\beta)},$$

at hinc valores nimis prodeunt complicati. Facilius negotium absolvetur resolvendo valores litterarum evanescentium B, C et E; nam

$$E=0 \quad \text{dat} \quad \zeta=\frac{\varepsilon\varepsilon}{\gamma}; \quad \text{tum} \quad B=0 \quad \text{dat} \quad \delta=\gamma+\frac{\alpha\varepsilon}{\beta}$$

atque

$$C=0 \quad \text{dat} \quad \delta\delta-\gamma\gamma=\alpha\zeta+2\beta\varepsilon=\frac{\alpha\varepsilon\varepsilon}{\gamma}+2\beta\varepsilon=\frac{\alpha^2\varepsilon\varepsilon}{\beta\beta}+\frac{2\alpha\gamma\varepsilon}{\beta},$$

cuius factores sunt $\beta\beta=\alpha\gamma$ et $\alpha\varepsilon\varepsilon+2\beta\gamma\varepsilon=0$. At si esset $\beta\beta=\alpha\gamma$, foret $A=0$; sin autem esset $\varepsilon=0$, foret et $\zeta=0$ et $D=0$, contra scopum. Fieri ergo oportet $\alpha\varepsilon=-2\beta\gamma$; unde fiet

$$\alpha=-\frac{2\beta\gamma}{\varepsilon}, \quad \delta=-\gamma \quad \text{et} \quad \zeta=\frac{\varepsilon\varepsilon}{\gamma}.$$

Denique fieri debet

$$\beta\beta+\frac{2\beta\gamma\gamma}{\varepsilon}=A \quad \text{et} \quad -2\gamma\varepsilon-\frac{\beta\varepsilon\varepsilon}{\gamma}=D.$$

Inde fit $\varepsilon=\frac{2\beta\gamma\gamma}{A-\beta\beta}$ et ob $\frac{\gamma D}{\varepsilon}=-(2\gamma\gamma+\beta\varepsilon)$ et $2\gamma\gamma+\beta\varepsilon=\frac{A\varepsilon}{\beta}$ erit $\frac{\gamma D}{\varepsilon}=-\frac{A\varepsilon}{\beta}$ ideoque $\varepsilon\varepsilon=-\frac{\beta\gamma D}{A}$. Ergo

$$\frac{4\beta\gamma^3}{(A-\beta\beta)^2}+\frac{D}{A}=0.$$

28. Cum autem tantum ratio litterarum A et D in censum veniat, aequatio ultima valori absoluto ipsius A inveniendo inservit, quem autem nosse non est opus. Manebunt ergo litterae γ et β indeterminatae. Ponatur ergo

$$\gamma=-Ac \quad \text{et} \quad \beta=Dc;$$

erit $\varepsilon\varepsilon=DDcc$ seu

$$\varepsilon=Dc \quad \text{hincque} \quad \delta=Ac, \quad \zeta=-\frac{DDc}{A} \quad \text{et} \quad \alpha=2Ac.$$

Quare huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+2Dx^3)}}=\frac{dy}{\sqrt{(A+2Dy^3)}}$$

integrale est

$$0 = 2A + 2D(x+y) - A(xx+yy) + 2Axy + 2Dxy(x+y) - \frac{DD}{A}xxyy.$$

Hoc autem integrale non est completum, tale autem reddetur ponendo $\gamma = -A$ et $\beta = Dcc$, unde fit $\varepsilon\varepsilon = DDcc$ et $\varepsilon = Dc$; porro erit $\delta = A$, $\zeta = -\frac{DDcc}{A}$, $\alpha = 2Ac$, ita ut integrale completum sit

$$0 = 2Ac + 2Dcc(x+y) - A(xx+yy) + 2Axy + 2Dcxy(x+y) - \frac{DDcc}{A}xxyy,$$

ubi c est constans ab arbitrio pendens; unde fit

$$y = \frac{Dcc + Ax + Dcxx \pm \sqrt{c\left(2A + \frac{DD}{A}c^3\right)(A + 2Dx^3)}}{A - 2Dcx + \frac{DDcc}{A}xx}.$$

29. Hic casus notari meretur, quo $A = 1$ et $D = \frac{1}{2}$, ut habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{(1+x^3)}} = \frac{dy}{\sqrt{(1+y^3)}},$$

ubi ad fractiones tollendas loco c scribatur $2c$, eritque integrale completum

$$0 = 4c + 4cc(x+y) - xx - yy + 2xy + 2cxy(x+y) - ccxxyy$$

seu

$$y = \frac{2cc + x + cxx \pm 2\sqrt{c(1+c^3)(1+x^3)}}{1 - 2cx + ccxx}.$$

Integralia ergo particularia erunt

I. si $c = 0$, $\quad y = x$;

II. si $c = \infty$, $\quad y = \frac{2 \pm 2\sqrt{(1+x^3)}}{xx}$;

III. si $c = -1$, $\quad y = \frac{2 + x - xx}{1 + 2x + xx} = \frac{2-x}{1+x}$.

30. Ex eodem principio, si in § 26 loco litterarum A, B, C, D, E eaedem per quantitatem quampiam p multiplicentur, nihilo minus aequatio differen-

tialis erit

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = \frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

invenieturque

$$p = \frac{BB\varepsilon\varepsilon - DD\beta\beta}{BBE - ADD} + 2\frac{(AD\varepsilon - BE\beta)(A\varepsilon\varepsilon - E\beta\beta)}{(B\varepsilon - D\beta)(BBE - ADD)} - \frac{C(A\varepsilon\varepsilon - E\beta\beta)}{BBE - ADD};$$

tum erit

$$\gamma = \frac{A\varepsilon\varepsilon - E\beta\beta}{B\varepsilon - D\beta}, \quad \alpha = \frac{\beta\beta - Ap}{\gamma}, \quad \zeta = \frac{\varepsilon\varepsilon - Ep}{\gamma} \quad \text{atque} \quad \delta = \gamma + \frac{\alpha\varepsilon + Bp}{\beta},$$

ita ut litterae β et ε maneant indeterminatae, fietque propterea aequatio integralis completa

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy,$$

unde fit

$$y = \frac{-\beta - \delta x - \varepsilon xx \pm \sqrt{p(A+2Bx+Cxx+2Dx^3+Ex^4)}}{\gamma + 2\varepsilon x + \zeta xx}.$$

31. Notandum denique est non solum hanc aequationem differentialem, cuius integrale completum modo exhibui, sed etiam hanc multo latius patentem

$$\frac{m\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = \frac{n\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

semper algebraice et quidem complete integrari posse, dummodo coefficientium m et n ratio fuerit rationalis; haec enim integratio simili modo instituitur, quo supra usus sum ad aequationem, quae mihi hic praecipue erat proposita, integrandam. Methodus autem, cuius hic specimina attuli, ita mihi videtur comparata, ut indolem eius diligentius excolendo ad insignes usus apta reddi queat, unde haud contemnenda commoda in Analysin sint redundatura.

32. Hic autem observo formulam § 26 assumtam latius extendendo eiusmodi differentialia inter se comparari posse, quae sint disparia, atque adeo

exemplum disparitatis (§ 22) allatum hoc modo obtineri posse, ita ut omnia, quae hactenus sunt tradita, in hac generali investigatione contineantur. Fingatur scilicet haec aequatio integralis

$$(1)\quad \alpha xxyy + 2\beta xxy + 2\gamma xyy + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \varkappa = 0,$$

ex qua fit

$$(2)\quad y = \frac{-\beta xx - \zeta x - \theta + \sqrt{((\beta xx + \zeta x + \theta)^2 - (\alpha xx + 2\gamma x + \varepsilon)(\delta xx + 2\eta x + \varkappa))}}{\alpha xx + 2\gamma x + \varepsilon},$$

$$(3)\quad x = \frac{-\gamma yy - \zeta y - \eta - \sqrt{((\gamma yy + \zeta y + \eta)^2 - (\alpha yy + 2\beta y + \delta)(\varepsilon yy + 2\theta y + \varkappa))}}{\alpha yy + 2\beta y + \delta}.$$

Ponatur iam brevitatis gratia

$$\begin{array}{ll|ll}
App &= \beta\beta - \alpha\delta & \mathfrak{A}qq &= \gamma\gamma - \alpha\varepsilon \\
2Bpp &= 2\beta\zeta - 2\alpha\eta - 2\gamma\delta & 2\mathfrak{B}qq &= 2\gamma\zeta - 2\alpha\theta - 2\beta\varepsilon \\
Cpp &= \zeta\zeta + 2\beta\theta - \alpha\varkappa - \delta\varepsilon - 4\gamma\eta & \mathfrak{C}qq &= \zeta\zeta + 2\gamma\eta - \alpha\varkappa - \delta\varepsilon - 4\beta\theta \\
2Dpp &= 2\zeta\theta - 2\gamma\varkappa - 2\varepsilon\eta & 2\mathfrak{D}qq &= 2\zeta\eta - 2\beta\varkappa - 2\delta\theta \\
Epp &= \theta\theta - \varepsilon\varkappa & \mathfrak{E}qq &= \eta\eta - \delta\varkappa
\end{array}$$

eritque

$$(4)\quad p\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)} = \alpha xxy + 2\gamma xy + \varepsilon y + \beta xx + \zeta x + \theta,$$

$$(5)\quad -q\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})} = \alpha xyy + 2\beta xy + \delta x + \gamma yy + \zeta y + \eta.$$

33. At si aequatio integralis assumta differentietur, fiet

$$(6)\quad \begin{aligned} &dx(\alpha xyy + 2\beta xy + \gamma yy + \delta x + \zeta y + \eta) \\ &+ dy(\alpha xxy + \beta xx + 2\gamma xy + \varepsilon y + \zeta x + \theta) = 0, \end{aligned}$$

unde, si istorum factorum valores (4) et (5) reperti substituantur, orietur ista aequatio differentialis

$$(7)\quad \frac{q\,dx}{\sqrt{(Ax^4 + 2Bx^3 + Cxx + 2Dx + E)}} = \frac{p\,dy}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})}},$$

cuius propterea integralis est aequatio assumta (1).

Cum autem supra habeantur 10 aequationes, coefficientium autem α, β, γ, δ etc. numerus sit 9, quorum unus pro lubitu assumi potest, octo remanebunt litterae determinandae. Porro autem insuper definiendae accedunt binae litterae p et q, ita ut nunc decem quantitates adsint incognitae, ex quo coefficientes utriusque formulae A, B, C, D, E et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ videntur pro lubitu assumi posse. Verum perspicuum est, cum alteri iam fuerint ad libitum assumti, alteros non omnino ab arbitrio nostro pendere; alias enim quaevis formula ad algebraicam reduci posset.

34. Hinc autem aliae datae formulae transmutationes non inelegantes obtineri possunt, si loco y alii valores substituantur. Veluti si ponatur $\mathfrak{E}=0$ seu $\eta\eta=\delta x$ statuaturque $y=zz$, sequens prodibit aequatio differentialis

$$\frac{q\,dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}=\frac{2p\,dz}{\sqrt{(\mathfrak{A}z^6+2\mathfrak{B}z^4+\mathfrak{C}z^2+2\mathfrak{D})}}, \tag{8}$$

cuius propterea integralis est aequatio assumta, si ponatur $y=zz$ statuaturque $\eta\eta=\delta x$ ac reliquae litterae rite determinentur. Integrale etiam completum nulla difficultate reperietur; nam etiamsi fortasse integrale inventum novam non involvat constantem, ponatur

$$\frac{q\,dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}=\frac{q\,du}{\sqrt{(Au^4+2Bu^3+Cuu+2Du+E)}}$$

et huius aequationis integrale completum ex antecedentibus assignare licebit atque hinc integrale quoque completum aequationis ex formulis disparibus constantis colligetur.

35. Quemadmodum huius aequationis differentialis, ut a simplicissimis incipiam,

$$\frac{dx}{\sqrt{(f+gx)}}=\frac{dy}{\sqrt{(f+gy)}}$$

integrale completum est

$$gg(xx+yy)-2ggxy-2ccg(x+y)+c^4-4ccf=0,$$

deinde vero huius aequationis differentialis

$$\frac{dx}{\sqrt{(f+gxx)}} = \frac{dy}{\sqrt{(f+gyy)}}$$

integrale completum est

$$xx + yy - 2xy\sqrt{(1+fgcc)} - ccff = 0,$$

tertio vero huius aequationis differentialis

$$\frac{dx}{\sqrt{(f+gx^3)}} = \frac{dy}{\sqrt{(f+gy^3)}}$$

integrale completum est

$$f(xx+yy) + \frac{ggcc}{4f}xxyy - gcxy(x+y) - 2fxy - gcc(x+y) - 2fc = 0,$$

quarto porro huius aequationis differentialis

$$\frac{dx}{\sqrt{(f+gx^4)}} = \frac{dy}{\sqrt{(f+gy^4)}}$$

integrale completum repertum est

$$f(xx+yy) - fcc - gccxxyy - 2xy\sqrt{f(f+gc^4)} = 0,$$

ita etiam integrale completum huius aequationis

$$\frac{dx}{\sqrt{(f+gx^6)}} = \frac{dy}{\sqrt{(f+gy^6)}}$$

reperiri poterit.

36. Determinentur primo in § 33 valores, ita ut prodeat haec aequatio

$$\frac{dx}{\sqrt{(fx+gx^4)}} = \frac{dy}{\sqrt{(fy+gy^4)}},$$

cuius integralis completa reperitur

$$gg(xx+yy) - 4ggcxxyy - 4fgccxy(x+y) - 2ggxy - 2fgc(x+y) + ffcc = 0.$$

Ponatur nunc $x=tt$ et $y=uu$, ut prodeat haec aequatio differentialis

$$\frac{dt}{\sqrt{(f+gt^6)}}=\frac{du}{\sqrt{(f+gu^6)}},$$

cuius propterea integralis completa erit

$$gg(t^4+u^4)-4ggct^4u^4-4fgccttuu(tt+uu)-2ggttuu-2fgc(tt+uu)+ffcc=0;$$

unde notari meretur casus ex hypothesi $c=\infty$ resultans, qui dat

$$4gttuu(tt+uu)=f.$$

OBSERVATIONES DE COMPARATIONE ARCUUM CURVARUM IRRECTIFICABILIUM

Commentatio 252 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1761, p. 58—84
Summarium ibidem p. 10—11

SUMMARIUM

Haec dissertatio ex eodem fonte est petita atque antecedens. Utraque enim innititur methodo formulas integrales, quae neque algebraice neque per angulos vel logarithmos expediri queant, algebraice inter se comparandi. Methodus autem ipsa, qua totum hoc negotium conficitur, ita est comparata, ut non data opera sit inventa, sed potius fortuito quasi detecta; ex quo, cum ad inventiones alias abstractissimas perduxerit, maxime digna videtur, ut omni studio uberius excolatur. In superiori quidem dissertatione hoc iam est praestitum, ut omnium curvarum, quarum arcus indefinite huiusmodi formula integrali $\int \frac{a\,dz}{\sqrt{(A+Bz+Cz^2+Dz^3+Ez^4)}}$ exprimuntur, arcus quicunque inter se comparari ac dato arcu quovis alii arcus ad eum datam rationem tenentes geometrice assignari queant, simili omnino modo, quo arcus circulares inter se comparari solent. Tali autem proprietate gaudet curva lemniscata vocari solita, cuius arcus indefinite hac formula $\int \frac{dz}{\sqrt{(1-z^4)}}$ exprimitur, huiusque arcuum comparatio in hac dissertatione prolixius explicatur. Praeterea vero Cel. Auctor investigationes suas ad arcus ellipticos et hyperbolicos extendit, in quo nova omnino vis illius methodi cernitur, cum rectificatio ellipsis et hyperbolae nullo modo ad formulam integralem ante commemoratam revocari possit. Neque vero etiam in his curvis comparatio arcuum uti in circulo institui potest; sed, quod iam pridem in arcubus parabolicis est factum, id nunc etiam istius novae methodi beneficio in ellipsi et hyperbola praestatur. Scilicet dato in altera curva arcu quocunque a puncto etiam dato semper alius arcus in eadem curva abscindi potest, cuius ab illo differentiam geometrice assignare liceat; tum vero etiam negotium ita confici potest, ut non ipsorum arcuum, sed quorumvis eorum

multiplorum differentia fiat geometrice assignabilis, idque ita, ut arcus quaesitus a dato puncto incipiat. Omissa autem hac conditione, ut arcus quaesitus in dato puncto terminetur, effici potest, ut differentia vel ipsorum arcuum vel quorumdam multiplorum eorundem evanescat sicque arcus assignari queant, qui absolute datam inter se teneant rationem. Atque hinc istud problema maxime notatu dignum resolvi potest, quo datus quicunque arcus, sive ellipticus sive hyperbolicus, ita secari iubetur, ut partium differentia geometrice assignabilis evadat. Sub finem animadvertit Auctor, quam insignia incrementa in Analysi nfinitorum hinc expectari queant, cum inde eiusmodi aequationum differentialium, quae nulli alii methodo cedant, integralia adeo algebraica assignari possint.

Speculationes mathematicae, si ad earum utilitatem respicimus, ad duas classes reduci debere videntur; ad priorem referendae sunt eae, quae cum ad vitam communem tum ad alias artes insigne aliquod commodum afferunt, quarum propterea pretium ex magnitudine huius commodi statui solet. Altera autem classis eas complectitur speculationes, quae, etsi cum nullo insigni commodo sunt coniunctae, tamen ita sunt comparatae, ut ad fines Analyseos promovendos viresque ingenii nostri acuendas occasionem praebeant. Cum enim plurimas investigationes, unde maxima utilitas expectari posset, ob solum analyseos defectum deserere cogamur, non minus pretium iis speculationibus statuendum videtur, quae haud contemnenda Analyseos incrementa pollicentur. Ad hunc autem scopum imprimis accommodatae videntur eiusmodi observationes, quae cum quasi casu sint factae et a posteriori detectae, ratio ad easdem a priori ac per viam directam perveniendi minus vel neutiquam est perspecta. Sic enim cognita iam veritate facilius in eas methodos inquirere licebit, quae ad eam directe sint perducturae, novis autem methodis investigandis Analyseos fines non mediocriter promoveri nullum plane est dubium.

Huiusmodi autem observationes, quae nulla certa methodo sunt factae quarumque ratio non parum abscondita videtur, nonnullas deprehendi in opere Ill. Comitis FAGNANI[1]) nuper in lucem edito; quae idcirco omni attentione dignae sunt censendae neque studium, quod in ulteriori earum investigatione consumitur, inutiliter collocatum erit iudicandum. Commemorantur autem in hoc libro quaedam eximiae proprietates, quibus curvae *Ellipsis*,

1) G. C. FAGNANO, *Produzioni matematiche*; vide notam p. 59. A. K.

Hyperbola et *Lemniscata* sunt praeditae, harumque curvarum arcus diversi inter se comparantur; cum igitur ratio harum proprietatum maxime occulta videatur, haud alienum fore arbitror, si eas diligentius examinavero, et quae mihi insuper circa has curvas elicere contigit, cum publico communicavero.

Quod igitur primum ad has curvas attinet, notum est earum rectificationem omnes Analyseos vires transcendere, ita ut earum arcus non solum non algebraice exprimi, sed etiam nequidem ad quadraturam circuli vel hyperbolae reduci queant. Quare eo magis mirum videri debet, quod Ill. Comes FAGNANO invenit, Ellipsi et Hyperbola infinitis modis eiusmodi binos arcus exhiberi posse, quorum differentia geometrice assignari queat, in curva lemniscata autem infinitis modis eiusmodi dari arcus binos, qui inter se vel sint aequales vel alter ad alterum rationem duplam teneat, unde deinceps modum colligit in hac curva etiam eiusmodi arcus assignandi, qui aliam inter se rationem teneant.

Pro Ellipsi quidem et Hyperbola nihil admodum mihi praeterea scrutari licuit; unde contentus ero faciliorem constructionem eorum arcuum dedisse, quorum differentia geometrice exhiberi queat. Pro curva autem lemniscata iisdem vestigiis insistens multo plures, imo infinitas elicui formulas, quarum beneficio non solum infinitis modis eiusmodi binos arcus definire possum, qui inter se vel sint aequales vel rationem teneant duplam, sed etiam qui sint inter se in ratione quacunque numeri ad numerum.

I. DE ELLIPSI

1. Sit quadrans ellipticus ABC (Fig. 1), cuius centrum in C, eiusque semiaxes ponantur $CA=1$ et $CB=c$; sumta ergo abscissa quacunque $CP=x$

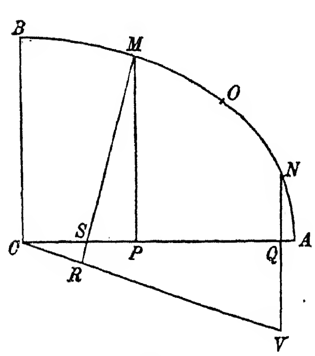

Fig. 1.

erit applicata ei respondens $PM=y=c\sqrt{(1-xx)}$; cuius differentiale cum sit $dy=-\frac{cxdx}{\sqrt{(1-xx)}}$, erit abscissae $CP=x$ arcus ellipticus respondens

$$BM=\int\frac{dx\sqrt{(1-(1-cc)xx)}}{\sqrt{(1-xx)}}.$$

Ponatur brevitatis gratia $1-cc=n$, ut sit arcus

$$BM=\int dx\sqrt{\frac{1-nxx}{1-xx}},$$

sumtaque alia quavis abscissa $CQ = u$ erit simili modo arcus ei respondens

$$BN = \int du \sqrt{\frac{1-nuu}{1-uu}}.$$

His positis quaeritur, quomodo hae duae abscissae x et u inter se comparatae esse debeant, ut arcuum summa

$$BM + BN = \int dx \sqrt{\frac{1-nxx}{1-xx}} + \int du \sqrt{\frac{1-nuu}{1-uu}}$$

integrabilis evadat seu geometrice exhiberi queat.

2. Quaestio ergo huc redit, ut determinetur, cuiusmodi functio ipsius x loco u substitui debeat, ut formula differentialis

$$dx \sqrt{\frac{1-nxx}{1-xx}} + du \sqrt{\frac{1-nuu}{1-uu}}$$

integrationem admittat. Facile autem perspicitur, si haec quaestio in genere consideretur, eius solutionem utriusque formulae integratione inniti ideoque aeque Analyseos fines transgredi atque ipsam ellipseos rectificationem. Cum igitur solutio generalis nullo modo expectari queat, in solutiones particulares erit inquirendum, quae uti nulla certa ratione reperiri possunt, ita etiam plurimum casui et coniecturae erit tribuendum; ex quo earum verum fundamentum, etiamsi ipsae sint cognitae[, vix poterit cognosci].

3. Primum quidem statim occurrit casus $u = -x$, quo formula nostra differentialis in nihilum abit; sed quia hinc duo Ellipseos arcus aequales et similes oriuntur, uti hic casus nimis est obvius, ita etiam quaestioni propositae minime satisfacere est censendus. Cum igitur tentaminibus totum negotium absolvi debeat, fingatur

$$\sqrt{\frac{1-nxx}{1-xx}} = \alpha u$$

et α ita concipiatur, ut vicissim fiat

$$\sqrt{\frac{1-nuu}{1-uu}} = \alpha x;$$

sic enim habebitur

$$BM + BN = \alpha \int u dx + \alpha \int x du = \alpha x u + \text{Const.},$$

omnino uti postulatur. Pro valore autem ipsius α habebimus tam

$$1 - nxx - \alpha\alpha uu + \alpha\alpha uuxx = 0 \quad \text{quam} \quad 1 - nuu - \alpha\alpha xx + \alpha\alpha xxuu = 0;$$

unde patet statui debere $\alpha\alpha = n$ et $\alpha = \sqrt{n}$, ita ut

$$u = \sqrt{\frac{1 - nxx}{n - nxx}} \quad \text{et} \quad BM + BN = xu\sqrt{n} + \text{Const.}$$

4. Etsi autem hoc modo quaestioni satisfactum videtur, tamen istae determinationes in Ellipsi locum habere nequeunt. Nam cum sit $n < 1$, quia $n = 1 - cc$, erit $n - nxx < 1 - nxx$ ideoque $u > 1$; abscissa ergo CQ semiaxem CA superaret eique propterea arcus imaginarius responderet, ita ut hinc nulla conclusio conformis deduci posset.

5. Tentemus ergo alias formulas sitque tam

$$\sqrt{\frac{1 - nxx}{1 - xx}} = \frac{\alpha}{u} \quad \text{quam} \quad \sqrt{\frac{1 - nuu}{1 - uu}} = \frac{\alpha}{x},$$

unde ob

$$\alpha\alpha - \alpha\alpha xx - uu + nxxuu = 0 \quad \text{et} \quad \alpha\alpha - \alpha\alpha uu - xx + nxxuu = 0$$

colligimus $\alpha = 1$, ita ut sit

$$1 - uu - xx + nxxuu = 0 \quad \text{ideoque} \quad u = \sqrt{\frac{1 - xx}{1 - nxx}}.$$

Hinc autem prodit

$$BM + BN = \int \frac{dx}{u} + \int \frac{du}{x} = \int \frac{xdx + udu}{xu}.$$

Verum aequatio $uu + xx = 1 + nxxuu$ differentiata dat

$$xdx + udu = nxu(xdu + udx) \quad \text{seu} \quad \frac{xdx + udu}{xu} = n(xdu + udx),$$

unde concludimus

$$BM + BN = n\int(xdu + udx) = nxu + \text{Const.}$$

6. Haec solutio nullo incommodo laborat; cum enim sit $n < 1$, erit $1 - nxx > 1 - xx$ ideoque $u < 1$, uti natura rei postulat. Sumta ergo ab-

scissa quacunque $CP = x$ capiatur altera

$$CQ = u = \sqrt{\frac{1-xx}{1-nxx}}$$

eritque summa arcuum $BM + BN = nxu + \text{Const.}$ Ad quam constantem definiendam sit $x = 0$, ut fiat $BM = 0$; eritque $u = 1$ et arcus BN abit in quadrantem $BMNA$; unde fit $0 + BMNA = 0 + \text{Const.}$ sicque haec constans erit $= BMNA$. Quo valore eius loco substituto habemus

$$BM + BN = nxu + BMNA$$

ideoque

$$BM - AN = nxu = (1-cc)xu = BN - AM.$$

7. Dato ergo in quadrante elliptico ACB puncto quocunque M assignare valemus alterum punctum N, ita ut differentia arcuum $BM - AN$, vel quae huic est aequalis $BN - AM$, geometrice exprimi queat. Quod quo facilius praestari possit, ducamus ad Ellipsin in puncto M normalem MS; erit subnormalis $PS = ccx$ et ob $PM = c\sqrt{(1-xx)}$ ipsa normalis

$$MS = c\sqrt{(1-xx+ccxx)} = c\sqrt{(1-nxx)};$$

ideoque pro altero puncto N abscissa erit $CQ = u = \frac{PM}{MS}CA$. Vel in normalem MS productam ex C demittatur perpendicularis CR, quae producatur in V, ut sit $CV = CA = 1$, et ob $\frac{CR}{CS} = \frac{PM}{MS}$ erit $CQ = \frac{CR}{CS}CV$. Quare ex puncto V in axem CA ducatur perpendicularis VQ, quae punctum Q et producta ipsum punctum N designabit.

8. Cum sit $PS = ccx$, erit $CS = x - ccx = nx$ ideoque

$$CR = \frac{CQ \cdot CS}{CV} = \frac{u \cdot nx}{1} = nux.$$

Hoc ergo ipsum perpendiculum CR differentiam arcuum $BM - AN$ seu $BN - AM$ exhibebit. Arcuum ergo hoc modo designatorum differentia erit $= nx\sqrt{\frac{1-xx}{1-nxx}}$, quae igitur evanescit tam casu $x = 0$ quam $x = 1$, quibus puncta M et N in ipsa puncta B et A incidunt. Maxima autem haec

differentia evadit, si $nx^4 - 2xx + 1 = 0$, hoc est si $x = \frac{1}{\sqrt{(1+c)}}$, quo casu fit $x = u$ et ambo puncta M et N in unum punctum O coeunt; eritque hoc casu differentia arcuum $BO - AO = nxx = 1 - c$ ideoque ipsi semiaxium differentiae $CA - CB$ fiet aequalis, ita ut sit $CA + AO = CB + BO$.

9. Si punctum M in ipso hoc puncto O capiatur, ut sit

$$CP = x = \frac{1}{\sqrt{(1+c)}},$$

erit

$$PM = \frac{c\sqrt{c}}{\sqrt{(1+c)}} \quad \text{et} \quad PS = \frac{cc}{\sqrt{(1+c)}}$$

hincque $MS = c\sqrt{c}$, unde variis modis situs puncti O commode definiri poterit. Cum autem sit

$$CM = CO = \frac{\sqrt{(1+c^3)}}{\sqrt{(1+c)}} = \sqrt{(1 - c + cc)} = \sqrt{(1 + cc - 2c\cos. 60^0)},$$

unde facilis constructio deducitur, sequentia ergo Theoremata subiungere visum est, quorum demonstratio ex allatis est manifesta.

THEOREMA 1

10. *In quadrante elliptico ACB* (Fig. 2) *si ad punctum quodvis M ducatur tangens HMK, quae cum altero axe CB in H concurrat, eaque alteri semiaxi CA aequalis capiatur, ut sit $HK = CA$, tum vero per K axi CB parallela agatur KN ellipsin secans in N, arcuum BM et AN differentia $BM - AN$ geometrice assignari poterit; demisso enim ex centro C in tangentem perpendiculo CT erit ista arcuum differentia $BM - AN = MT$.*

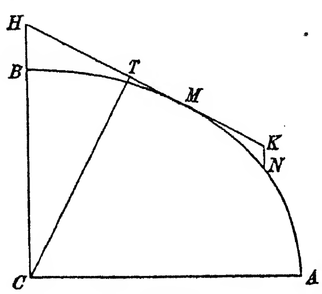

Fig. 2.

Demonstratio ex figura sponte patet, cum tangens HMK sit rectae illi CRV (Fig. 1, p. 82) parallela et aequalis; tum vero perspicuum est esse $MT = CR$.

THEOREMA 2

11. *Si super quadrantis elliptici ACB* (Fig. 3) *altero semiaxe CA triangulum aequilaterum CAE constituatur et in eius latere AE portio capiatur $AF = CB$ iunctaeque CF aequalis applicetur in ellipsi recta CO, punctum O hanc habebit proprietatem, ut sit*

$$CA + arcu\, AO = CB + arcu\, BO.$$

Demonstratio ex § 9 evidens est. Cum enim sit

$$CA = 1, \quad AF = c \quad \text{et} \quad \text{ang. } CAF = 60^0,$$

erit

$$CF = \sqrt{(1 + cc - 2c \cos. 60^0)}$$

ideoque $= CO$.

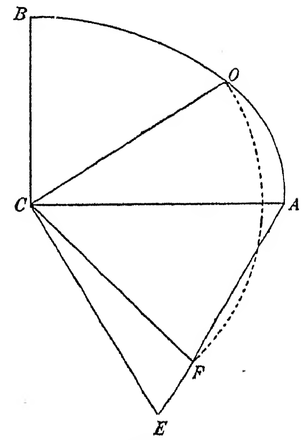

Fig. 3.

II. DE HYPERBOLA

12. Sit C (Fig. 4) centrum hyperbolae AMN eiusque semiaxis transversus $CA = 1$, semiaxis coniugatus $= c$; erit sumta abscissa quacunque $CP = x$ applicata $PM = c\sqrt{(xx - 1)}$ eiusque differentiale $= \frac{cxdx}{\sqrt{(xx-1)}}$; unde fit arcus

$$AM = \int \frac{dx\sqrt{((1+cc)xx - 1)}}{\sqrt{(xx-1)}}.$$

Ponatur brevitatis gratia $1 + cc = n$; erit

$$AM = \int dx \sqrt{\frac{nxx-1}{xx-1}}.$$

Simili ergo modo si capiatur alia quaevis abscissa $CQ = u$, erit arcus ei respondens

$$AN = \int du \sqrt{\frac{nuu-1}{uu-1}}.$$

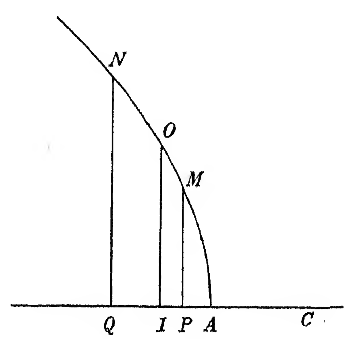

Fig. 4.

13. His positis ista nobis proposita sit quaestio, ut dato puncto M alterum N ita definiatur, ut summa arcuum $AM + AN$ seu expressio

$$\int dx \sqrt{\frac{nxx-1}{xx-1}} + \int du \sqrt{\frac{nuu-1}{uu-1}}$$

absolute integrationem admittat; quod quidem evenire casu $u = -x$ sponte patet; verum hinc nihil ad institutum nostrum concludere licet.

14. Ponamus ergo

$$\sqrt{\frac{nxx-1}{xx-1}} = u\sqrt{n},$$

cum hinc vicissim fiat

$$\sqrt{\frac{nuu-1}{uu-1}} = x\sqrt{n};$$

utrinque enim prodit haec aequatio $nuuxx - n(uu + xx) + 1 = 0$. Facta autem hac hypothesi prodit summa arcuum

$$AM + AN = \int udx\sqrt{n} + \int xdu\sqrt{n} = ux\sqrt{n} + \text{Const.}$$

Haec ergo integrabilitas ut locum habeat, oportet sit $u = \sqrt{\frac{nxx-1}{nxx-n}}$, unde, cum ob $n > 1$ prodeat quoque $u > 1$, ex dato puncto M semper alterum punctum N assignari poterit.

15. Ad constantem definiendam patet casum $x = 1$, quo punctum M in verticem A incidit, nihil iuvare, cum inde oriatur $u = \infty$ punctumque N in infinitum removeatur. Quocirca ut haec constans debite determinetur, alium casum considerari oportet; potior autem non occurrit quam is, ubi puncta M et N in unum coalescunt seu quo fit $u = x$ et $nx^4 - 2nxx + 1 = 0$. Hinc autem oritur

$$xx = 1 + \frac{c}{\sqrt{(1+cc)}} \quad \text{et} \quad x = \sqrt{\left(1 + \frac{c}{\sqrt{(1+cc)}}\right)}.$$

Sit igitur O hoc punctum, in quo ambo puncta M et N coalescunt, , OI erit abscissa

$$\frac{c}{\sqrt{(1+cc)}}\Big) \quad \text{et} \quad 2AO = c + \sqrt{(1+cc)} + \text{Const.}$$

Hinc ergo obtinemus constantem quaesitam

$$=2AO-c-\sqrt{(1+cc)}$$

ob $\sqrt{n}=\sqrt{(1+cc)}$. Quo valore substituto erit pro quibusvis punctis M et N diversis ita sumtis, ut sit $u=\sqrt{\frac{nxx-1}{nxx-n}}$, summa arcuum

$$AM+AN=ux\sqrt{n}+2AO-c-\sqrt{(1+cc)}$$

seu

$$ON-OM=ux\sqrt{n}-c-\sqrt{(1+cc)}.$$

Sic igitur duos arcus nacti sumus ON et OM, quorum differentia $ON-OM$ geometrice assignari potest.

17. Quo autem facilius pateat, quomodo tam punctum O quam ex puncto M punctum N definiri queat, erigatur in A (Fig. 5) perpendiculum $AD=c$ eritque recta CD hyperbolae asymtota; tum positis $CP=x$, $PM=y$ ducatur tangens MT; erit ob

$$y=c\sqrt{(xx-1)} \quad \text{et} \quad dy=\frac{cxdx}{\sqrt{(xx-1)}}$$

subtangens

$$PT=\frac{y\sqrt{(xx-1)}}{cx}=x-\frac{1}{x} \quad \text{et} \quad CT=\frac{1}{x}$$

et ipsa tangens

$$MT=\frac{y\sqrt{(nxx-1)}}{cx}.$$

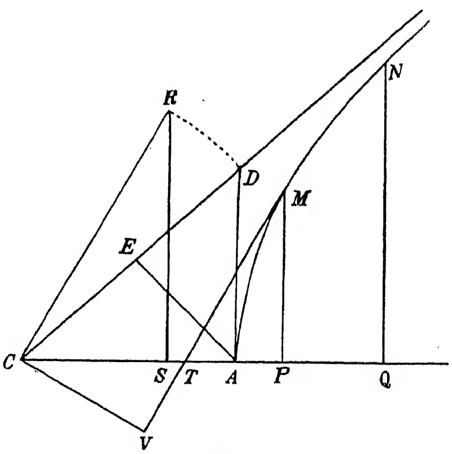

Fig. 5.

Hinc prodit

$$\sqrt{\frac{xx-1}{nxx-1}}=\frac{PT}{MT} \quad \text{ideoque} \quad u=\frac{MT}{PT\sqrt{(1+cc)}}=\frac{CA^2\cdot MT}{CD\cdot PT}=CQ.$$

18. Ducatur ex centro C tangenti TM parallela $CR=CD$ demissoque ex R in axem perpendiculo RS erit $CS=\frac{CD\cdot PT}{MT}$ ideoque $CQ=\frac{CA^2}{CS}$. Quare CQ capienda erit tertia proportionalis ad CS et CA. Commodius

autem res sequenti modo sine tangentium adminiculo expedietur; nam cum sit

$$QN = \frac{cc}{\sqrt{n(xx-1)}} = \frac{c^3}{y\sqrt{n}},$$

erit

$$PM \cdot QN = \frac{c^3}{\sqrt{(1+cc)}} = \frac{AD^3}{CD}$$

vel demisso ex A in asymtotam perpendiculo AE erit

$$PM \cdot QN = AD \cdot DE$$

ob $DE = \frac{AD^2}{CD}$, unde sequens Theorema conficitur.

THEOREMA 3

19. *Existente AOZ* (Fig. 6) *hyperbola, C eius centro, A vertice et CDZ eius asymtota, ad quam ex A axi perpendiculariter ducta sit recta AD itemque AE ad asymtotam perpendicularis, si applicata constituatur IO media proportionalis inter AD et DE atque utrinque applicatae PM et QN ita statuantur, ut inter eas sit IO media proportionalis, tum arcuum ON et OM differentia geometrice assignari poterit. Erit enim*

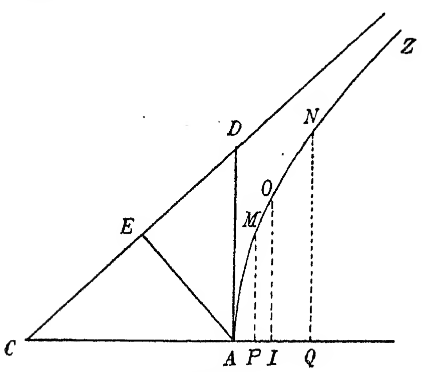

Fig. 6.

$$ON - OM = \frac{CP \cdot CQ - CI \cdot CI}{CE}.$$

Demonstratio ex paragrapho praecedente est manifesta. Cum enim punctis M et N in O coeuntibus sit $IO \cdot IO = AD \cdot DE$, erit IO media proportionalis inter AD et DE; hacque inventa esse oportet $PM \cdot QN = OI \cdot OI$. Tum vero ex § 16 intelligitur esse $ON - OM = (CP \cdot CQ - CI \cdot CI)\sqrt{n}$ et ob $\sqrt{n} = CD$ erit homogeneitatem implendo $ON - OM = (CP \cdot CQ - CI \cdot CI)\frac{CD}{CA^2}$. At est $\frac{CA^2}{CD} = CE$ sicque constat Theorematis veritas.

III. DE CURVA LEMNISCATA

20. Haec curva ob plurimas, quibus praedita est, insignes proprietates inter Geometras est celebrata, imprimis autem, quod eius arcus arcubus curvae

elasticae sunt aequales. Natura autem huius curvae ita est comparata, ut positis coordinatis orthogonalibus $CP = x$, $PM = y$ (Fig. 7) ista aequatione exprimatur

$$(xx + yy)^2 = xx - yy.$$

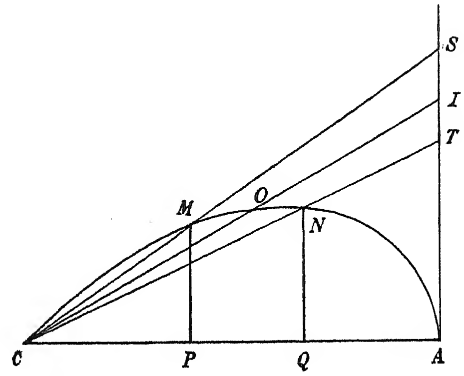

Fig. 7.

Unde patet hanc curvam esse lineam quarti ordinis, quae in C, quod punctum eius centrum dicitur, cum axe CA angulum semirectum constituit, in A autem sumta $CA = 1$ axem normaliter traiicit. Figura autem $CMNA$ quartam partem totius lemniscatae exhibet, cui tres reliquae partes circa centrum C aequales sunt concipiendae; id quod inde liquet, quod, sive abscissa x sive applicata y sive utraque negativum valorem induat, aequatio eadem manet.

21. Quod igitur ad expressionem arcus cuiusque CM huius curvae attinet, is commodissime ex corda CM definitur. Si enim hanc cordam ponamus $CM = z$, ob $xx + yy = zz$ habebimus $z^4 = xx - yy = 2xx - zz = zz - 2yy$, unde elicimus

$$x = z\sqrt{\frac{1+zz}{2}} \quad \text{et} \quad y = z\sqrt{\frac{1-zz}{2}}$$

et differentiando

$$dx = \frac{dz(1+2zz)}{\sqrt{2(1+zz)}} \quad \text{et} \quad dy = \frac{dz(1-2zz)}{\sqrt{2(1-zz)}}.$$

Hinc ergo elementum arcus CM colligitur

$$\sqrt{(dx^2 + dy^2)} = dz\sqrt{\frac{(1-zz)(1+2zz)^2 + (1+zz)(1-2zz)^2}{2(1+zz)(1-zz)}}$$

sive

$$\sqrt{(dx^2 + dy^2)} = \frac{dz}{\sqrt{(1-z^4)}}.$$

22. Si ergo corda quaecunque ex centro C educta ponatur $CM = z$, erit arcus ab ea subtensus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$. Simili ergo modo si alia quaevis corda CN dicatur $= u$, erit arcus ab ea subtensus $CN = \int \frac{du}{\sqrt{(1-u^4)}}$, cuius

complementum ad totum quadrantem est arcus AN. Iam Ill. Comes FAGNANO docuit, cuiusmodi functio ipsius z capi debeat pro u, ut vel arcus AN aequalis fiat arcui CM, vel ut arcus CN sit duplus arcus CM, vel etiam ut arcus AN sit aequalis duplo arcui CM. Hos ergo casus primo exponam, deinceps autem, quae mihi circa alias huiusmodi arcuum proportiones eruere contigit, in medium sum allaturus.

THEOREMA 4

23. *In curva lemniscata hactenus descripta si applicetur corda quaecunque* $CM = z$ *aliaque insuper applicetur, quae sit*

$$CN = u = \sqrt{\frac{1-zz}{1+zz}},$$

erit arcus CM *aequalis arcui* AN *vel etiam arcus* CN *aequalis arcui* AM.

DEMONSTRATIO

Cum sit corda $CM = z$, erit arcus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$ et ob cordam $CN = u$ erit arcus $CN = \int \frac{du}{\sqrt{(1-u^4)}}$. At est $u = \sqrt{\frac{1-zz}{1+zz}}$; unde fit

$$du = \frac{-2zdz}{(1+zz)\sqrt{(1-z^4)}}.$$

Praeterea vero est

$$u^4 = \frac{1-2zz+z^4}{1+2zz+z^4} \quad \text{ideoque} \quad 1-u^4 = \frac{4zz}{(1+zz)^2} \quad \text{et} \quad \sqrt{(1-u^4)} = \frac{2z}{1+zz}.$$

Quibus valoribus substitutis habebitur

$$\text{arc. } CN = -\int \frac{dz}{\sqrt{(1-z^4)}} = -\text{arc. } CM + \text{Const.},$$

ita ut sit arc. $CN +$ arc. $CM =$ Const. Ad hanc constantem definiendam perpendatur casus, quo $z = 0$ ideoque et arcus $CM = 0$; hoc autem casu fit $u = 1 = CA$ ideoque arcus CN abit in quadrantem $CMNA$, ex ur pro hoc casu $CMNA + 0 =$ Const. Hoc ergo valore substituto

prodibit in genere arc. $CN+$ arc. $CM=$ arc. $CMNA$ hincque

$$\text{arc. } CM = \text{arc. } AN$$

et arcum MN utrinque addendo

$$\text{arc. } CMN = \text{arc. } ANM.$$

Q. E. D.

COROLLARIUM 1

24. Dato ergo quocunque arcu CM in centro C terminato, cuius corda est $CM = z$, ei ab altera parte seu vertice A abscindetur arcus aequalis AN sumendo cordam

$$CN = u = \sqrt{\frac{1-zz}{1+zz}} \quad \text{seu} \quad CN = CA\sqrt{\frac{CA^2 - CM^2}{CA^2 + CM^2}}$$

homogeneitatem supplendo per axem $CA = 1$.

COROLLARIUM 2

25. Cum sit $u = \sqrt{\frac{1-zz}{1+zz}}$, erit vicissim $z = \sqrt{\frac{1-uu}{1+uu}}$; unde cordas CM et CN inter se permutare licet, ita ut, si ambae cordae $CM = z$ et $CN = u$ ita fuerint comparatae, ut sit

$$uuzz + uu + zz = 1,$$

etiam puncta M et N inter se permutari queant indeque prodeat tam arc. $CM=$ arc. AN quam arc. $CN=$ arc. AM.

COROLLARIUM 3

26. Cum sit $CN = u = \sqrt{\frac{1-zz}{1+zz}}$, erit

$$\sqrt{\frac{1+uu}{2}} = \frac{1}{\sqrt{(1+zz)}} \quad \text{et} \quad \sqrt{\frac{1-uu}{2}} = \frac{z}{\sqrt{(1+zz)}}.$$

Unde, cum ex natura curvae lemniscatae pro puncto N coordinatae sint

$$CQ = u\sqrt{\frac{1+uu}{2}} \quad \text{et} \quad QN = u\sqrt{\frac{1-uu}{2}},$$

erit

$$CQ = \frac{u}{\sqrt{(1+zz)}} \quad \text{et} \quad QN = \frac{uz}{\sqrt{(1+zz)}} \quad \text{ideoque} \quad \frac{QN}{CQ} = z.$$

Quare si in A ad axem CA erigatur normalis AT, donec cordae CN productae occurrat in T, erit $AT = z = CM$.

COROLLARIUM 4

27. Ex dato ergo puncto M alterum punctum N ita facillime definitur: capiatur tangens AT aequalis cordae CM ductaque recta CT curvam in puncto quaesito N secabit. Ob eandem autem rationem patet: si corda CM producatur, donec tangenti in A occurrat in S, erit pariter $AS = CN$.

COROLLARIUM 5

28. Manifestum etiam est puncta M et N in unum punctum O coire posse, in quo propterea totus quadrans COA in duas partes aequales dividitur. Invenietur ergo hoc punctum O, si ponatur $u = z$, unde fit

$$z^4 + 2zz = 1 \quad \text{hincque} \quad zz + 1 = \sqrt{2};$$

prodit ergo corda $CO = \sqrt{(\sqrt{2} - 1)}$, cui simul tangens AI erit aequalis, unde simul positio huius puncti O facile assignatur.

COROLLARIUM 6

29. Notato ergo hoc puncto O, quo totus quadrans COA in duas partes aequales CMO et ANO dividitur, erit quoque punctis M et N per regulam expositam definitis arc. MO = arc. ON, ita ut idem hoc punctum O omnes arcus MN in duas partes aequales dispescat.

THEOREMA 5

30. *In curva lemniscata, cuius axis* $CA = 1$ (Fig. 8), *si applicata sit corda quaecunque* $CM = z$ *aliaque insuper chorda applicetur*

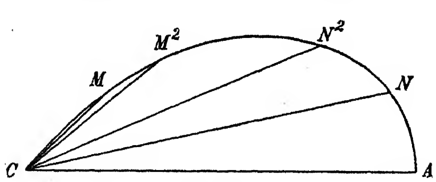

Fig. 8.

$$CM^2 = u = \frac{2z\sqrt{(1-z^4)}}{1+z^4},$$

erit arcus a corda hac u subtensus CM^2 *duplo maior quam arcus ab illa corda subtensus* CM.

DEMONSTRATIO

Cum sit corda $CM = z$, erit arcus $CM = \int \frac{dz}{\sqrt{(1-z^4)}}$ similiterque ob cordam $CM^2 = u$ erit arcus $CM^2 = \int \frac{du}{\sqrt{(1-u^4)}}$. Quia autem est $u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$, erit

$$uu = \frac{4zz - 4z^6}{1+2z^4+z^8}$$

ideoque

$$\sqrt{(1-uu)} = \frac{1-2zz-z^4}{1+z^4} \quad \text{et} \quad \sqrt{(1+uu)} = \frac{1+2zz-z^4}{1+z^4},$$

unde fit

$$\sqrt{(1-u^4)} = \frac{1-6z^4+z^8}{(1+z^4)^2}.$$

Tum vero differentiando colligitur

$$du = \frac{2dz(1-z^8) - 4z^4dz(1+z^4) - 8z^4dz(1-z^4)}{(1+z^4)^2\sqrt{(1-z^4)}}$$

seu

$$du = \frac{2dz - 12z^4dz + 2z^8dz}{(1+z^4)^2\sqrt{(1-z^4)}} = \frac{2dz(1-6z^4+z^8)}{(1+z^4)^2\sqrt{(1-z^4)}}.$$

Hinc ergo nanciscimur

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{2dz}{\sqrt{(1-z^4)}}$$

et integrando arc. $CM^2 = 2$ arc. $CM +$ Const. Cum autem posito $z = 0$ fiat etiam $u = 0$ ideoque ambo arcus CM et CM^2 evanescant, constans quoque in nihilum abit. Sicque sumta corda $CM^2 = u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$ erit

$$\text{arcus } CM^2 = 2 \text{ arc. } CM.$$

Q. E. D.

COROLLARIUM 1

31. Si capiatur corda $CN = \sqrt{\frac{1-zz}{1+zz}}$, erit arcus $AN =$ arc. CM hincque etiam arcus CM^2 erit $= 2$ arc. AN. Simili modo si capiatur corda $CN^2 = \sqrt{\frac{1-uu}{1+uu}}$, erit arcus $AN^2 =$ arc. CM^2 sicque etiam a vertice A erit arc. $AN^2 = 2$ arc. AN. Hoc ergo modo obtinentur quatuor arcus inter se aequales, scilicet arc. CM, arc. MM^2, arc. AN et arc. NN^2.

COROLLARIUM 2

32. Cum autem sit

$$u=\frac{2z\sqrt{(1-z^4)}}{1+z^4},\quad \sqrt{(1-uu)}=\frac{1-2zz-z^4}{1+z^4}\quad \text{et}\quad \sqrt{(1+uu)}=\frac{1+2zz-z^4}{1+z^4},$$

hae quatuor cordae ita habebuntur expressae, ut sit

$$CM=z,\quad CN=\sqrt{\frac{1-zz}{1+zz}},\quad CM^2=\frac{2z\sqrt{(1-z^4)}}{1+z^4},\quad CN^2=\frac{1-2zz-z^4}{1+2zz-z^4}.$$

COROLLARIUM 3

33. Conveniant ambo puncta M^2 et N^2 in curvae puncto medio O (Fig. 9), pro quo supra vidimus esse cordam $CO=\sqrt{(\sqrt{2}-1)}$, atque hoc casu tota curva COA in quatuor partes aequales dispescetur in punctis M, O et N. Hoc igitur evenit, si sit $CM^2=CN^2=\sqrt{(\sqrt{2}-1)}$, ita ut posito brevitatis gratia $\sqrt{(\sqrt{2}-1)}=\alpha$ habeamus

$$1-2zz-z^4=\alpha+2\alpha zz-\alpha z^4\quad \text{seu}\quad z^4=\frac{-2(1+\alpha)zz+1-\alpha}{1-\alpha}$$

et

$$zz=\frac{-(1+\alpha)+\sqrt{2(1+\alpha\alpha)}}{1-\alpha}\quad \text{vel}\quad zz=\frac{-1-\sqrt{(\sqrt{2}-1)}+\sqrt{2}\sqrt{2}}{1-\sqrt{(\sqrt{2}-1)}}.$$

Unde colligimus

$$CM=z=\sqrt{\frac{-1-\alpha+\sqrt{2(1+\alpha\alpha)}}{1-\alpha}}\quad \text{et}\quad CN=\sqrt{\frac{-1+\alpha+\sqrt{2(1+\alpha\alpha)}}{1+\alpha}}.$$

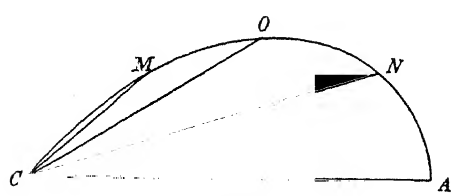

Fig. 9.

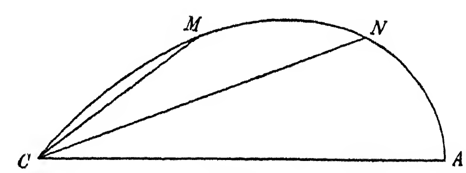

Fig. 10.

COROLLARIUM 4

34. Coalescant ambo puncta M^2 et N (Fig. 10) et puncta M et N^2 pariter coibunt sicque tota curva $CMNA$ in punctis M et N trifariam secabitur. Pro hoc ergo casu habebitur vel

$$\frac{2z\sqrt{(1-z^4)}}{1+z^4}=\sqrt{\frac{1-zz}{1+zz}}\quad \text{vel}\quad z=\frac{1-2zz-z^4}{1+2zz-z^4},$$

quarum posterior dat $1-z-2zz-2z^3-z^4+z^5=0$ haecque per $1+z$ divisa $1-2z-2z^3+z^4=0$; cuius concipiantur factores

$$(1-\mu z+zz)(1-\nu z+zz)=0$$

eritque $\mu+\nu=2$ et $\mu\nu=-2$, unde fit $\mu-\nu=2\sqrt{3}$ hincque

$$\mu=1+\sqrt{3} \quad \text{et} \quad \nu=1-\sqrt{3}.$$

Erit ergo

$$z=\frac{1+\sqrt{3}\pm\sqrt{2\sqrt{3}}}{2}=CM$$

et ob $zz=\frac{4+4\sqrt{3}\pm 2(1+\sqrt{3})\sqrt{2\sqrt{3}}}{4}$ orietur

$$CN=\sqrt{\frac{1-zz}{1+zz}}=\sqrt{\frac{-2\sqrt{3}\mp(1+\sqrt{3})\sqrt{2\sqrt{3}}}{4+2\sqrt{3}\pm(1+\sqrt{3})\sqrt{2\sqrt{3}}}}=\sqrt{\frac{\mp\sqrt{2\sqrt{3}}}{1+\sqrt{3}}}.$$

Est itaque

$$CM=\frac{1+\sqrt{3}-\sqrt{2\sqrt{3}}}{2} \quad \text{et} \quad CN=\sqrt{\frac{\sqrt{2\sqrt{3}}}{1+\sqrt{3}}}.$$

COROLLARIUM 5

35. Dato etiam quocunque arcu CM^2 (Fig. 8, p. 94) inveniri potest eius semissis CM; si enim arcus illius ponatur corda $CM^2=u$ et arcus quaesiti corda $CM=z$, erit

$$u=\frac{2z\sqrt{(1-z^4)}}{1+z^4} \quad \text{et} \quad 1-\frac{4zz}{uu}+2z^4+\frac{4z^6}{uu}+z^8=0,$$

cuius factores concipiantur

$$(1-\mu zz-z^4)(1-\nu zz-z^4)=0;$$

unde obtinetur $\mu+\nu=\frac{4}{uu}$ et $\mu\nu=4$; erit ergo

$$\mu-\nu=4\sqrt{\left(\frac{1}{u^4}-1\right)}=\frac{4}{uu}\sqrt{(1-u^4)}$$

hincque

$$\mu=\frac{2+2\sqrt{(1-u^4)}}{uu} \quad \text{et} \quad \nu=\frac{2-2\sqrt{(1-u^4)}}{uu},$$

ergo

$$zz = \frac{-1-\sqrt{(1-u^4)}+\sqrt{2(1+\sqrt{(1-u^4)})}}{uu};$$

unde pro z duplex valor realis elicitur, alter

$$z = \frac{\sqrt{\left(-1-\sqrt{(1-u^4)}+\sqrt{2(1+\sqrt{(1-u^4)})}\right)}}{u} = \frac{\sqrt{(1-\sqrt{(1-uu)})(\sqrt{(1+uu)}-1)}}{u},$$

alter

$$z = \frac{\sqrt{\left(-1+\sqrt{(1-u^4)}+\sqrt{2(1-\sqrt{(1-u^4)})}\right)}}{u} = \frac{\sqrt{(1+\sqrt{(1-uu)})(\sqrt{(1+uu)}-1)}}{u}.$$

COROLLARIUM 6

36. Duplex hic valor revera locum obtinet; cum enim eadem corda CM^2 (Fig. 11) et Cm^2 duos arcus diversos CM^2 et CM^2m^2 subtendat, alter valor ipsius z praebebit cordam arcus CM, qui est semissis arcus CM^2, alter autem valor ipsius z dat cordam arcus Cm, qui est semissis arcus CM^2m^2; ac prior quidem valor pro illo casu, posterior vero pro hoc locum habet.

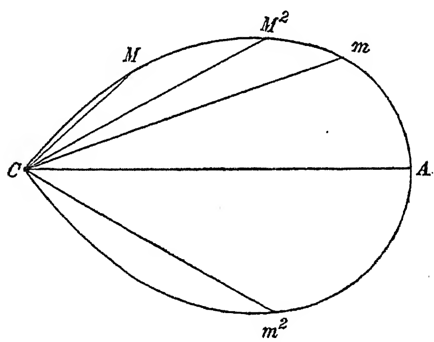

Fig. 11.

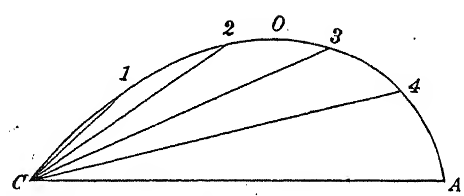

Fig. 12.

COROLLARIUM 7

37. Hoc modo etiam lemniscata CA in quinque partes aequales dividi potest. Sit enim corda partis simplicis $C1 = z$ (Fig. 12), corda partis duplicatae

$$C2 = \frac{2z\sqrt{(1-z^4)}}{1+z^4} = u;$$

erit corda partis quadruplicatae

$$C4 = \frac{2u\sqrt{(1-u^4)}}{1+u^4} = \sqrt{\frac{1-zz}{1+zz}},$$

quia est $A4 = C1$, unde corda z definitur; qua inventa, cum sit $C2 = A3$, erit corda $C3 = \sqrt{\frac{1-uu}{1+uu}}$.

COROLLARIUM 8

38. Cum hinc posita corda cuiuspiam $= z$ reperiri possint cordae arcuum dupli, quadrupli, octupli, sedecupli etc., manifestum est hoc modo etiam lemniscatam in tot partes dividi posse, quarum numerus sit $2^m(1+2^n)$. In hac autem formula continentur sequentes numeri

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 16,\ 17,\ 18,\ 20,\ 24,\ 32,\ 33 \text{ etc.}$$

Verum hinc non semper omnia divisionum puncta assignare licet.

SCHOLION

39. Haec igitur sunt, quae Ill. Comes FAGNANO de curva lemniscata observavit vel quae ex eius inventis derivare licet. Etsi enim tantum proposito arcu quocunque eius duplum assignare docuit, tamen hunc arcum iterum continuo duplicando etiam cordae arcuum quadrupli, octupli, sedecupli etc. inde colligentur. Namque si corda arcus simpli statuatur $= z$, arcus dupli $= u$, quadrupli $= p$, octupli $= q$, sedecupli $= r$ etc., erit

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4}$$

$$p = \frac{2u\sqrt{(1-u^4)}}{1+u^4} = \frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{(1-z^4)}}{(1+z^4)^4+16z^4(1-z^4)^2}$$

$$q = \frac{2p\sqrt{(1-p^4)}}{1+p^4}$$

$$r = \frac{2q\sqrt{(1-q^4)}}{1+q^4} \quad \text{etc.}$$

Aliorum autem arcuum multiplorum cordas ex his assignare non licet. Quemadmodum ergo arcuum quorumvis multiplorum cordae exprimantur, hic investigabo, ut hoc argumentum, quantum limites Analyseos id quidem permittunt, penitus perficiatur. Primum quidem tentando elicui, si arcus simpli corda sit $= z$, tum arcus tripli cordam fore $= \frac{z(3-6z^4-z^8)}{1+6z^4-3z^8}$; verum postea rem sequenti modo generaliter expediri posse intellexi.

THEOREMA 6

40. *Si corda arcus simplicis* CM (Fig. 13) *sit* $=z$ *et corda arcus* n*-cupli* $CM^n=u$, *erit corda arcus* $(n+1)$*-cupli*

$$CM^{n+1}=\frac{z\sqrt{\frac{1-uu}{1+uu}}+u\sqrt{\frac{1-zz}{1+zz}}}{1-uz\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}}.$$

DEMONSTRATIO

Erit ergo ipse arcus simplex

$$CM=\int\frac{dz}{\sqrt{(1-z^4)}}$$

et arcus n-cuplus

$$CM^n=\int\frac{du}{\sqrt{(1-u^4)}}=n\int\frac{dz}{\sqrt{(1-z^4)}}$$

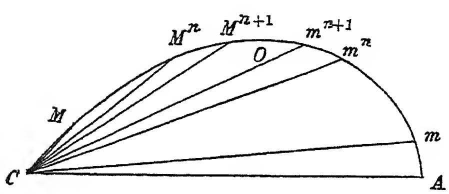

Fig. 13.

ideoque habemus $du=\frac{n\,dz\sqrt{(1-u^4)}}{\sqrt{(1-z^4)}}$. Ponamus brevitatis gratia

$$z\sqrt{\frac{1-uu}{1+uu}}=P \quad \text{et} \quad u\sqrt{\frac{1-zz}{1+zz}}=Q,$$

ut sit corda pro arcu $(n+1)$-cuplo exhibita $CM^{n+1}=\frac{P+Q}{1-PQ}$, quae dicatur $=s$, atque demonstrari oportet esse arcum huic cordae respondentem

$$\int\frac{ds}{\sqrt{(1-s^4)}}=(n+1)\int\frac{dz}{\sqrt{(1-z^4)}} \quad \text{seu} \quad \frac{ds}{\sqrt{(1-s^4)}}=\frac{(n+1)dz}{\sqrt{(1-z^4)}}.$$

Cum autem sit $s=\frac{P+Q}{1-PQ}$, erit

$$ds=\frac{dP(1+QQ)+dQ(1+PP)}{(1-PQ)^2};$$

tum vero reperitur

$$1-s^4=\frac{(1-PQ)^4-(P+Q)^4}{(1-PQ)^4}=\frac{(1+PP+QQ+PPQQ)(1-PP-QQ-4PQ+PPQQ)}{(1-PQ)^4}$$

ergo

$$\sqrt{(1-s^4)}=\frac{\sqrt{(1+PP)(1+QQ)(1-PP-QQ-4PQ+PPQQ)}}{(1-PQ)^2},$$

ex quo elicitur

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP\sqrt{\frac{1+QQ}{1+PP}}+dQ\sqrt{\frac{1+PP}{1+QQ}}}{\sqrt{(1-PP-QQ-4PQ+PPQQ)}},$$

cuius expressionis ergo valorem investigemus.

Ac primo quidem est

$$1+PP=\frac{1+uu+zz-uuzz}{1+uu} \quad \text{et} \quad 1+QQ=\frac{1+uu+zz-uuzz}{1+zz},$$

ita ut sit $\frac{1+PP}{1+QQ}=\frac{1+zz}{1+uu}$ ideoque

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP\sqrt{\frac{1+uu}{1+zz}}+dQ\sqrt{\frac{1+zz}{1+uu}}}{\sqrt{(1-PP-QQ+PPQQ-4PQ)}}.$$

Deinde vero ob

$$1-PP=\frac{1+uu-zz+uuzz}{1+uu} \quad \text{et} \quad 1-QQ=\frac{1+zz-uu+uuzz}{1+zz}$$

erit

$$(1-PP)(1-QQ)=1-P^2-Q^2+P^2Q^2=\frac{1-z^4-u^4+4uuzz+u^4z^4}{(1+zz)(1+uu)}$$

et

$$4PQ=\frac{4uz\sqrt{(1-z^4)(1-u^4)}}{(1+zz)(1+uu)};$$

hincque concluditur denominator

$$\sqrt{(1-PP-QQ+PPQQ-4PQ)}$$
$$=\frac{\sqrt{(1-z^4-u^4+4uuzz+u^4z^4-4uz\sqrt{(1-z^4)(1-u^4)})}}{\sqrt{(1+zz)(1+uu)}}=\frac{\sqrt{(1-z^4)(1-u^4)}-2uz}{\sqrt{(1+zz)(1+uu)}},$$

ex quo obtinebitur

$$\frac{ds}{\sqrt{(1-s^4)}}=\frac{dP(1+uu)+dQ(1+zz)}{\sqrt{(1-z^4)(1-u^4)}-2uz}.$$

Iam vero differentiando elicimus

$$dP=dz\sqrt{\frac{1-uu}{1+uu}}-\frac{2zudu}{(1+uu)\sqrt{(1-u^4)}},$$
$$dQ=du\sqrt{\frac{1-zz}{1+zz}}-\frac{2zudz}{(1+zz)\sqrt{(1-z^4)}},$$

quare ob

$$du = \frac{ndz\sqrt{(1-u^4)}}{\sqrt{(1-z^4)}}$$

erit

$$dP = dz\sqrt{\frac{1-uu}{1+uu}} - \frac{2nuzdz}{(1+uu)\sqrt{(1-z^4)}},$$

$$dQ = \frac{ndz\sqrt{(1-u^4)}}{1+zz} - \frac{2uzdz}{(1+zz)\sqrt{(1-z^4)}},$$

unde conficitur numerator

$$dP(1+uu) + dQ(1+zz) = dz\sqrt{(1-u^4)} - \frac{2nuzdz}{\sqrt{(1-z^4)}} + ndz\sqrt{(1-u^4)} - \frac{2uzdz}{\sqrt{(1-z^4)}}$$

sive

$$dP(1+uu) + dQ(1+zz) = (n+1)dz\sqrt{(1-u^4)} - \frac{2(n+1)uzdz}{\sqrt{(1-z^4)}}$$
$$= \frac{(n+1)dz}{\sqrt{(1-z^4)}}\left(\sqrt{(1-z^4)(1-u^4)} - 2uz\right),$$

unde perspicuum est esse

$$\frac{ds}{\sqrt{(1-s^4)}} = \frac{(n+1)dz}{\sqrt{(1-z^4)}}$$

et

$$\text{arc. } CM^{n+1} = (n+1) \text{ arc. } CM.$$

Q. E. D.

COROLLARIUM 1

41. Si a vertice A abscindantur arcus Am, Am^n, Am^{n+1} arcubus CM, CM^n, CM^{n+1} respective aequales, erit Cm corda complementi arcus CM, Cm^n corda complementi arcus CM^n, Cm^{n+1} corda complementi arcus CM^{n+1}. Erunt autem ob cordas $CM = z$, $CM^n = u$, $CM^{n+1} = s$ complementorum cordae

$$Cm = \sqrt{\frac{1-zz}{1+zz}}, \quad Cm^n = \sqrt{\frac{1-uu}{1+uu}}, \quad Cm^{n+1} = \sqrt{\frac{1-ss}{1+ss}}.$$

Cum autem sit

$$s = \frac{z\sqrt{\frac{1-uu}{1+uu}} + u\sqrt{\frac{1-zz}{1+zz}}}{1 - zu\sqrt{\frac{(1-uu)(1-zz)}{(1+uu)(1+zz)}}} = \frac{P+Q}{1-PQ},$$

erit

$$\sqrt{\frac{1-ss}{1+ss}} = \sqrt{\frac{1-PP-QQ-4PQ+PPQQ}{(1+PP)(1+QQ)}} = \frac{\sqrt{(1-z^4)(1-u^4)} - 2uz}{1+uu+zz-uuzz},$$

quae ad hanc formam reducitur

$$\sqrt{\frac{1-ss}{1+ss}} = \frac{\sqrt{\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}} - uz}{1 + uz\sqrt{\frac{(1-zz)(1-uu)}{(1+zz)(1+uu)}}}.$$

COROLLARIUM 2

42. Si igitur ponatur

corda arcus simplicis $= z$, corda complementi $= Z$,
corda arcus n-cupli $= u$, corda complementi $= U$,

ut sit

$$Z = \sqrt{\frac{1-zz}{1+zz}} \quad \text{et} \quad U = \sqrt{\frac{1-uu}{1+uu}},$$

erit

$$\text{corda arcus } (n+1)\text{-cupli} = \frac{zU + uZ}{1 - zuZU},$$

$$\text{corda complementi} = \frac{ZU - zu}{1 + zuZU}.$$

COROLLARIUM 3

43. Inventio ergo cordarum arcuum quorumvis multiplorum una cum cordis complementi ita se habebit:

Corda arcus	Corda complementi
simpli $= a$	simpli $= A$
dupli $= b = \dfrac{2aA}{1-aaAA}$	dupli $= \dfrac{AA-aa}{1+aaAA} = B$
tripli $= c = \dfrac{aB+bA}{1-abAB}$	tripli $= \dfrac{AB-ab}{1+abAB} = C$
quadrupli $= d = \dfrac{aC+cA}{1-acAC}$	quadrupli $= \dfrac{AC-ac}{1+acAC} = D$
quintupli $= e = \dfrac{aD+dA}{1-adAD}$	quintupli $= \dfrac{AD-ad}{1+adAD} = E$
etc.	etc.

COROLLARIUM 4

44. Simili modo si corda arcus m-cupli sit $= r$, corda complementi $= R$ et corda arcus n-cupli $= s$ eiusque corda complementi $= S$, ut sit

$$R = \sqrt{\frac{1-rr}{1+rr}} \quad \text{et} \quad S = \sqrt{\frac{1-ss}{1+ss}},$$

erit corda arcus $(m+n)$-cupli $=\frac{rS+sR}{1-rsRS}$ et corda complementi $=\frac{RS-rs}{1+rsRS}$. Quin etiam sumendo pro n numerum negativum, quia tum corda s abit in sui negativum, corda differentiae illorum arcuum exhiberi poterit; erit scilicet corda arcus $(m-n)$-cupli $=\frac{rS-sR}{1+rsRS}$ et corda complementi eius $=\frac{RS+rs}{1-rsRS}$.

COROLLARIUM 5

45. Sumtis ergo denominationibus, quae in corollario 3 sunt adhibitae, erit quoque

$$d=\frac{2bB}{1-bbBB} \quad \text{et} \quad D=\frac{BB-bb}{1+bbBB}$$

$$e=\frac{bC+cB}{1-bcBC} \quad \text{et} \quad E=\frac{BC-bc}{1+bcBC}.$$

COROLLARIUM 6

46. Ex his colligitur, si corda arcus simplicis statuatur $=z$, valores cordarum in corollario 3 adhibitarum fore

$$a=z \qquad A=\sqrt{\frac{1-zz}{1+zz}}$$

$$b=\frac{2z\sqrt{(1-z^4)}}{1+z^4} \qquad B=\frac{1-2zz-z^4}{1+2zz-z^4}$$

$$c=\frac{z(3-6z^4-z^8)}{1+6z^4-3z^8} \qquad C=\frac{(1+z^4)^2-4zz(1+zz)^2}{(1+z^4)^2+4zz(1-zz)^2}\sqrt{\frac{1-zz}{1+zz}}$$

$$d=\frac{4z(1+z^4)(1-6z^4+z^8)\sqrt{(1-z^4)}}{(1+z^4)^4+16z^4(1-z^4)^2} \qquad D=\frac{(1-6z^4+z^8)^2-8zz(1-z^4)(1+z^4)^2}{(1-6z^4+z^8)^2+8zz(1-z^4)(1+z^4)^2}.$$

SCHOLION 1

47. Ratio compositionis formularum $\frac{rS+sR}{1-rsRS}$ et $\frac{RS-rs}{1+rsRS}$ imprimis ideo notari meretur, quod similis est regulae, qua tangens summae vel differentiae duorum angulorum definiri solet. Si enim sit $rS=\text{tang.}\,\alpha$ et $sR=\text{tang.}\,\beta$, erit $\frac{rS+sR}{1-rsRS}=\text{tang.}\,(\alpha+\beta)$ et pro differentia in corollario 4 exhibita $\frac{rS-sR}{1+rsRS}=\text{tang.}\,(\alpha-\beta)$. Similique modo si ponatur $RS=\text{tang.}\,\gamma$ et $rs=\text{tang.}\,\delta$, erit

$$\frac{RS-rs}{1+rsRS}=\text{tang.}\,(\gamma-\delta) \quad \text{et} \quad \frac{RS+rs}{1-rsRS}=\text{tang.}\,(\gamma+\delta).$$

Commodius autem ista compositionis ratio repraesentabitur, si ponatur corda arcus m-cupli $r = M \sin. \mu$, corda complementi $R = M \cos. \mu$, corda arcus n-cupli $s = N \sin. \nu$, corda complementi $S = N \cos. \nu$; tum enim erit

$$\begin{aligned}
&\text{corda arcus } (m+n)\text{-cupli} &&= \frac{MN \sin. (\mu + \nu)}{1 - M^2 N^2 \sin. \mu \sin. \nu \cos. \mu \cos. \nu}\\
&\text{corda eius complementi} &&= \frac{MN \cos. (\mu + \nu)}{1 + M^2 N^2 \sin. \mu \sin. \nu \cos. \mu \cos. \nu}\\
&\text{corda arcus } (m-n)\text{-cupli} &&= \frac{MN \sin. (\mu - \nu)}{1 + M^2 N^2 \sin. \mu \sin. \nu \cos. \mu \cos. \nu}\\
&\text{corda eius complementi} &&= \frac{MN \cos. (\mu - \nu)}{1 - M^2 N^2 \sin. \mu \sin. \nu \cos. \mu \cos. \nu}.
\end{aligned}$$

Cum autem sit $1 - rr - RR = rrRR$, erit $1 - MM = M^4 \sin. \mu^2 \cos. \mu^2$ ideoque

$$M^2 \sin. \mu \cos. \mu = \sqrt{(1 - MM)} \quad \text{et} \quad N^2 \sin. \nu \cos. \nu = \sqrt{(1 - NN)},$$

unde istarum formularum denominatores abibunt in

$$1 - \sqrt{(1 - MM)(1 - NN)} \quad \text{et} \quad 1 + \sqrt{(1 - MM)(1 - NN)}.$$

Praeterea vero ex illa aequatione $1 - MM = M^4 \sin. \mu^2 \cos. \mu^2$ fit

$$\frac{1}{MM} = \frac{1}{2} + \frac{1}{2}\sqrt{(1 + \sin. 2\mu \sin. 2\mu)}$$

ob $\sin. 2\mu = 2 \sin. \mu \cos. \mu$. Verum hinc illae formulae non concinniores evadunt.

SCHOLION 2

48. Ex his observationibus calculus integralis non contemnenda augmenta consequitur, siquidem hinc plurimarum aequationum differentialium integrales particulares exhibere valemus, quarum integratio in genere vix sperari potest. Sic proposita aequatione differentiali

$$\frac{du}{\sqrt{(1 - u^4)}} = \frac{dz}{\sqrt{(1 - z^4)}},$$

praeterquam quod casus integralis $u = z$ per se est obvius, novimus ei quoque satisfacere $u = -\sqrt{\frac{1 - zz}{1 + zz}}$. In genere igitur cum integratio constantem arbitrariam, puta C, involvat, erit u aequalis functioni cuipiam quantitatum z et C; quae tamen nihilominus ita erit comparata, ut pro certo quodam ipsius C valore fiat $u = z$ itemque pro alio quodam ipsius C valore $u = -\sqrt{\frac{1 - zz}{1 + zz}}$.

Duo ergo dantur valores, qui constanti huic C tributi functionem illam in expressionem algebraicam adeo simplicem convertunt.

Simili modo proposita hac aequatione

$$\frac{du}{\sqrt{(1-u^4)}} = \frac{2dz}{\sqrt{(1-z^4)}}$$

duos habemus valores, quos ei satisfacere novimus,

$$u = \frac{2z\sqrt{(1-z^4)}}{1+z^4} \quad \text{et} \quad u = \frac{-1+2zz+z^4}{1+2zz-z^4}$$

pariterque geminos valores exhibere docuimus, qui in genere huic aequationi satisfaciant

$$\frac{m\,du}{\sqrt{(1-u^4)}} = \frac{n\,dz}{\sqrt{(1-z^4)}},$$

unde via ad harum formularum integralia generalia invenienda non parum praeparata videtur.

Deinde quae supra de ellipsi et hyperbola sunt allata, sequentes aequationum differentialium integrationes speciales suppeditant.

Proposita enim ex § 3 hac aequatione

$$dx\sqrt{\frac{1-nxx}{1-xx}} + du\sqrt{\frac{1-nuu}{1-uu}} = (xdu+udx)\sqrt{n}$$

novimus ei satisfacere hanc aequationem integralem

$$1-nxx-nuu+nuuxx=0.$$

Isti autem aequationi ex § 5 petitae

$$dx\sqrt{\frac{1-nxx}{1-xx}} + du\sqrt{\frac{1-nuu}{1-uu}} = n(xdu+udx)$$

satisfacere inventa est haec aequatio

$$1-xx-uu+nuuxx=0.$$

Deinde sequenti aequationi ex hyperbola § 14 petitae

$$dx\sqrt{\frac{nxx-1}{xx-1}} + du\sqrt{\frac{nuu-1}{uu-1}} = (xdu+udx)\sqrt{n}$$

$$1-nxx-nuu+nuuxx=0,$$

quae quidem cum priore ex ellipsi petita congruit, cum sit

$$\sqrt{\frac{nxx-1}{xx-1}}=\sqrt{\frac{1-nxx}{1-xx}}.$$

Hinc autem facile concludere licet, huic aequationi

$$dx\sqrt{\frac{f-gxx}{h-kxx}}+du\sqrt{\frac{f-guu}{h-kuu}}=(xdu+udx)\sqrt{\frac{g}{h}}$$

satisfacere hanc integralem specialem

$$fh-gh(xx+uu)+gkxxuu=0,$$

isti autem aequationi alteri

$$dx\sqrt{\frac{f-gxx}{h-kxx}}+du\sqrt{\frac{f-guu}{h-kuu}}=(xdu+udx)\frac{g}{\sqrt{fk}}$$

satisfacere hanc integralem specialem

$$fh-fk(xx+uu)+gkxxuu=0.$$

Haec igitur ideo proponenda censui, quod ansam mihi praebere videntur subsidia Analyseos ulterius excolendi.

SPECIMEN NOVAE METHODI CURVARUM QUADRATURAS ET RECTIFICATIONES ALIASQUE QUANTITATES TRANSCENDENTES INTER SE COMPARANDI

Commentatio 263 indicis Enestroemiani
Novi Commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p. 83—127
Summarium (Commentationum 263 et 261) ibidem p. 5—8

SUMMARIUM

Principio monendus est lector rogandaque errori typothetarum excusatio est, quod posterior ordine dissertatio[1]) priori est anteposita. Culpam hanc aliquodam modo resarcituri utramque dissertationem simul considerabimus et consueta nobis brevitate, quid in iis praestitum sit, dicemus. Versatur methodus a Cel. Auctore proposita singulari prorsus modo circa quantitates transcendentes seu eiusmodi quantitates in lineis curvis occurrentes, quae nullo modo algebraice exprimi possunt. Semper consideratio linearum, utcunque sterilis in se videatur, tam Geometriam quam Analysin pulcerrimis inventis locupletavit. Cum primum enim Geometrae lineas curvas contemplari coeperunt, statim omnibus viribus eo sunt annisi, ut tam spatia ab iis inclusa quam ipsam earum longitudinem dimetirentur, quarum investigationum prior circa curvarum quadraturas, altera circa earum rectificationem versari dicebatur. Quoniam vero neutrum in circulo praestari poterat, etsi omnium linearum curvarum est simplicissima, eo maiori studio in eiusmodi lineas curvas inquisiverunt, quae vel quadraturam, hoc est spatii iis inclusi dimensionem, vel rectificationem, qua linea recta curvae aequalis assignari debet, admitterent. Interim tamen etiam inutiles conatus eorum, qui in quadratura circuli investiganda frustra desudarunt, praeter opinionem plurima egregia inventa sunt consecuti, quibus idem usu venit, quod Alchimistis, qui toti in lapidis philo-

1) L. Euleri Commentatio 261 (indicis Enestroemiani); vide p. 153. A. K.

sophorum praeparatione occupati, etsi voto suo exciderunt, plurima saluberrima remedia in usum medicinae contulerunt. Post inventam autem Analysin infinitorum summum studium, quod praecipue in quadrandis et rectificandis lineis curvis est consumtum, uberrimos fructus protulit, quibus plures methodos satis sublimes, quarum usus per universam Mathesin amplissimus existit, acceptas referre debemus. Quare haud minores fructus ab eorum studio expectare licet, qui in comparatione linearum curvarum, quae per se vel quadraturam vel rectificationem respuunt, exquirenda laborant, in quo negotio certe profundissima Analyseos arcana sunt adeunda, ita ut, qui hic quicquam praestiterit, is plurimum in hac scientia profecisse sit censendus.

Huc sine dubio referenda est nova methodus a Cel. Auctore excogitata, cuius ope innumerabilium curvarum, quarum rectificatio omnes vires Analyseos transcendit, arcus inter se comparare docet. Pro iis quidem curvis, quarum rectificatio ope circuli vel logarithmorum expediri potest, hoc cognitis methodis praestari potest, sed totum negotium multo facilius beneficio huius methodi conficitur, quemadmodum ex specimine posteriore luculenter apparet, ubi comparationem arcuum circularium, aliunde quidem satis cognitam, et arcuum parabolicorum mira simplicitate exequitur, ut iam hinc summa utilitas huius methodi abunde eluceat.

In altero autem specimine, quod hic primo loco extat, hanc methodum potissimum ad Ellipsin accommodatam conspicimus, cuius lineae rectificationem neque ad arcus circulares neque logarithmos revocari posse inter Geometras satis superque constat. Neque etiam in hac curva binos arcus dissimiles, qui inter se sint aequales, abscindere licet, ex quo multo magis mirum videbitur dato huius curvae arcu quocunque semper alium arcum et in dato quidem puncto terminatum exhiberi posse, qui ab illo differat quantitate geometrice assignabili, cum hoc ne in circulo quidem praestari queat. Si enim differentia inter duos arcus circulares geometrice assignari posset, eo ipso rectificatio circuli absoluta haberetur. In ellipsi autem haec ratio longe aliter est comparata, cum innumerabilibus modis differentia in binos arcus ellipticos definiri possit. Simili modo, proposito arcu ellipseos quocunque, ab alio quovis puncto arcum abscindere licet, qui ab illius duplo vel triplo vel alio quovis multiplo atque etiam submultiplo quantitate geometrice assignabili differat. Imo etiam fieri potest, ut haec differentia prorsus evanescat sicque bini arcus elliptici datam inter se rationem tenentes exhiberi queant, dummodo ratio illa non sit aequalitatis, quippe quo casu bini arcus prodeunt inter se similes, in quo nihil singulare habetur. Cuncta autem haec problemata, quae Cel. Auctor hic pro Ellipsi expedivit, simili plane modo etiam pro Hyperbola atque infinitis aliis lineis curvis multo magis complicatis resolvi posse manifestum est; ex quo haec methodus omni Geometrarum attentione et uberiori evolutione dignissima videtur.

Quae nuper[1]) occasione inventorum Ill. Comitis FAGNANI commentatus sum de comparatione arcuum ellipsis, hyperbolae et curvae lemniscatae, multo latius mihi quidem patere statim sunt visa. Cum enim methodis adhuc consuetis eiusmodi tantum curvarum arcus inter se comparari possent, quarum rectificatio vel a quadratura circuli vel a logarithmis penderet, quippe quae quantitates, etsi sunt transcendentes, tamen ita iam in Analysi prae ceteris ius quoddam civitatis sunt adeptae, ut perinde atque algebraicae tractari queant, maxima certe attentione erat dignum, quod a FAGNANO in hyperbola et ellipsi arcus sint assignati, quorum differentia sit algebraica, in lemniscata autem eiusmodi arcus, qui adeo inter se sint aequales vel certam teneant rationem, propterea quod harum curvarum rectificatio neque ad quadraturam circuli neque ad logarithmos reduci queat. Hinc certe theoriae quantitatum transcendentium insigne lumen accenderetur, si modo via, qua FAGNANUS est usus, certam methodum suppeditaret in huiusmodi investigationibus ulterius progrediendi; sed quia tota substitutionibus precario factis et quasi casu fortuito adhibitis nititur, parum inde utilitatis in Analysin redundat. Deinde iam notavi integrationes, quas operatio FAGNANIANA complectitur, tantum esse particulares neque idcirco methodum certam, a qua plura expectari liceat, suppeditare. Interim tamen ea amplissimum campum aperuisse est visa, in quo ulterius excolendo Geometrae vires suas summo cum fructu exerceant ad insigne Analyseos incrementum.

Res autem huc redit, ut propositis duabus formulis integralibus $\int X dx$ et $\int Y dy$ non integrabilibus, ubi X sit functio quaepiam ipsius x et Y ipsius y, eiusmodi relatio inter variabiles x et y definiatur, ut illae formulae vel inter se fiant aequales vel datam rationem teneant, vel ut differentiam algebraice assignabilem obtineant. Quae investigatio cum latissime pateat, tum etiam insignes in se continet casus iam pridem non sine maximo Analyseos incremento evolutos; huc enim referenda sunt, quae de comparatione arcuum circularium, de lunulis quadrabilibus, de zonis cycloidalibus quadrabilibus, tum vero de arcubus parabolicis, qui vel datam inter se teneant rationem vel differentiam algebraicam habeant, a geometris sunt tradita; quin etiam haec investigatio a Cel. IOH. BERNOULLI[2]) ad parabolas cubicales altiorisque ordinis

1) L. EULERI Commentatio 252 (indicis ENESTROEMIANI); vide p. 80. A. K.

2) IOH. BERNOULLI, *Investigatio algebraica arcuum parabolicorum assignatam inter se rationem habentium. Demonstratio isochronismi descensuum in cycloide* etc., Acta erud. 1698, p. 261; *Opera omnia* T. 1, p. 242; *Theorema universale rectificationi linearum curvarum inserviens. Nova parabolarum proprietas. Cubicalis primariae arcuum mensura* etc., Acta erud. 1698, p. 462; *Opera omnia* T. 1, p. 249. A. K.

est extensa, sed quia ratio, qua usus est, nulla certa methodo nitebatur, ulteriori usu fere penitus caruit. Hoc quoque pertinet, quod multo ante iam acutissimus HUGENIUS[1]) in *Horologio oscillatorio* exposuerat, ubi proposito sphaeroide elliptico compresso seu revolutione circa axem minorem genito invenire docuit conoides hyperbolicum, ita ut summa utriusque superficiei circulo exhiberi posset, cum tamen neutra superficies seorsim cum circulo comparari queat. Quae inventio iam tum summis Geometris maxime memorabilis visa est; atque BERNOULLIUS in litteris ad LEIBNIZIUM[2]) datis dolet hanc inventionem nulla certa methodo inniti, ex qua plura huius generis inventa derivare liceat; interim quia superficies tam illius sphaeroidis elliptici quam conoidis hyperbolici a logarithmis pendet, reductio utriusque iunctim sumtae ad circulum simili modo perfici potest, quo in parabola arcus algebraicam habentes differentiam assignari solent. Inprimis autem hoc loco non est praetereundum TSCHIRNHAUSIUM[3]) quondam methodum a se inventam iactasse, cuius beneficio curvarum quarumcunque non rectificabilium arcus ita inter se comparare possent, ut differentia fiat algebraica; sed praeterquam, quod methodum suam nunquam aperuerit, manifestum est eum paralogismo quodam fuisse deceptum ut saepius alias, cum certum sit rem ita generaliter omnino expediri non posse; neque ergo TSCHIRNHAUSIUS putandus est quicquam eorum habuisse, quae vel tum circa comparationem curvarum sunt inventa vel adhuc forte elicientur.

Specimen igitur quoddam methodi huiusmodi quaestiones solvendi hic exhibere constitui, quod non obscure maiores progressus in hac re promittere videtur; atque cum non solum difficillimum sit propositis in genere eiusmodi formulis integralibus quaesitam inter variabiles relationem eruere, sed etiam hoc saepissime omnino ne fieri quidem possit, ordine inverso rem ita tentavi, ut assumta binarum variabilium relatione inde ipsas formulas integrales investigarem, quae per hanc relationem inter se comparari possent. Quae methodus cum facillime procedat, ad multo sublimiora perducere posse videtur, quae aliis methodis plane sint impervia; hac enim methodo non

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1) CHR. HUYGENS (1629—1695), *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae*, Parisiis 1673; *Opera varia* Vol. 1, 1724, p. 15, imprimis p. 105. A. K.

2) Hic EULERUS errasse videtur; cf. IOH. BERNOULLI, *Meditatio de dimensione linearum curvarum per circulares,* Acta erud. 1695, p. 374; *Opera omnia* T. 1, p. 142. A. K.

3) W. TSCHIRNHAUS (1651—1708), *Nova et singularis geometriae promotio circa dimensionem quantitatum curvarum*, Acta erud. 1695, p. 489. A. K.

solum ea, quae habet FAGNANUS, facili negotio ac sine taedioso calculo sum assecutus, sed etiam multo ampliora atque illustriora reddidi, ut, quae ille nimis particulariter definiverat, ego satis universaliter expediverim; atque calculus, quo sum usus, ita comparatus est, ut, quoniam operationes prorsus singulares complectitur, viam ad multo sublimiora sternere videatur.

Tum vero quanquam variabilium mutua relatio per methodos consuetas definiri potest, quoties integratio utriusque formulae $\int X dx$ et $\int Y dy$ vel a quadratura circuli vel a logarithmis pendet, tamen et hoc plerumque non sine molesto calculo perficitur, dum partes vel arcus circulares vel logarithmos continentes se mutuo destruere debent, quemadmodum hoc in comparatione arcuum parabolicorum abunde perspicitur. Per meam autem methodum hae difficultates cunctae penitus evanescunt ac fere sine ullo negotio istae comparationes tam in circulo quam in parabola absolvuntur; in quo sine dubio non exigua vis huius methodi sita esse censenda est, quod non solum multo facilius ea, quae aliis methodis iam sunt eruta, praebeat, sed etiam ad eiusmodi investigationes manuducat, in quibus aliae methodi nihil essent praestiturae. Quam ob rem hoc quidem loco istam methodum tantum ad eos casus applicabo, qui etiam aliis methodis, sed multo operosius, expediri solent, quo, cum principia, quibus innititur, hac occasione exposuero, deinceps facilius eius applicationem ad quaestiones sublimiores suscipere possim. Quoniam igitur mihi a relatione inter binas variabiles, quam pro lubitu constituo, ordiendum est, a simplicioribus incipiam ac primo quidem ab eiusmodi, quae ad similes formulas integrales perducant, seu in quibus X et Y similes sint proditurae functiones ipsarum x et y. Formulae ergo integrales hinc natae ob similitudinem quantitates transcendentes exhibebunt ad eandem lineam curvam pertinentes, deinceps autem ad formulas quoque dissimiles, quae ad diversas curvas pertineant, sum progressurus.

RELATIO PRIMA INTER BINAS VARIABILES x ET y

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy$$

1. Si hinc seorsim valores x et y extrahantur, reperietur

$$y = \frac{-\delta x \pm \sqrt{((\delta\delta - \gamma\gamma)xx - \alpha\gamma)}}{\gamma},$$

$$x = \frac{-\delta y \pm \sqrt{((\delta\delta - \gamma\gamma)yy - \alpha\gamma)}}{\gamma},$$

ibi quovis casu dispiciendum est, utrum signum quantitati radicali sit prae-

figendum. Fieri enim potest, ut in utraque formula vel signa paria vel disparia locum habeant, dum alterutrum arbitrio nostro plane relinquitur; in quo iudicio inprimis natura variabilium x et y, utrum affirmative an negative accipiantur, spectari debet.

2. Ponantur brevitatis gratia membra irrationalia

$$\pm\sqrt{((\delta\delta-\gamma\gamma)xx-\alpha\gamma)}=P \quad \text{et} \quad \pm\sqrt{((\delta\delta-\gamma\gamma)yy-\alpha\gamma)}=Q,$$

ut sit

$$y=\frac{-\delta x+P}{\gamma} \quad \text{et} \quad x=\frac{-\delta y+Q}{\gamma},$$

sicque erit

$$P=\gamma y+\delta x \quad \text{et} \quad Q=\gamma x+\delta y,$$

unde quovis casu facile colligere licet, utrum quantitates P et Q habiturae sint valores affirmativos an negativos.

3. Differentietur iam aequatio assumta eritque

$$dx(\gamma x+\delta y)+dy(\gamma y+\delta x)=0$$

atque ob $\gamma x+\delta y=Q$ et $\gamma y+\delta x=P$ habebitur haec aequatio

$$Qdx+Pdy=0 \quad \text{sive} \quad \frac{dx}{P}+\frac{dy}{Q}=0.$$

Restitutis ergo pro P et Q valoribus huic aequationi integrali

$$\int\frac{dx}{\sqrt{((\delta\delta-\gamma\gamma)xx-\alpha\gamma)}}+\int\frac{dy}{\sqrt{((\delta\delta-\gamma\gamma)yy-\alpha\gamma)}}=\text{Const.}$$

satisfacit relatio inter variabiles x et y assumta.

4. Evolvamus haec accuratius, et quo facilius applicatio fieri queat, ponamus

$$-\alpha\gamma=Ap \quad \text{et} \quad \delta\delta-\gamma\gamma=Cp,$$

ut sit

$$\int\frac{dx}{\sqrt{(A+Cxx)}}+\int\frac{dy}{\sqrt{(A+Cyy)}}=\text{Const.},$$

eritque

$$\alpha = -\frac{Ap}{\gamma} \quad \text{et} \quad \delta = \sqrt{(Cp + \gamma\gamma)}$$

sicque quantitates p et γ arbitrio nostro relinquuntur.

5. Statuatur ergo $\gamma = A$ et $p = Akk$, ita ut k sit nova quantitas constans a nostro arbitrio pendens, eritque

$$\alpha = -Akk, \quad \gamma = A \quad \text{et} \quad \delta = \sqrt{A(A + Ckk)}$$

et aequatio canonica nostrae aequationi integrali satisfaciens erit

$$0 = -Akk + A(xx + yy) + 2xy\sqrt{A(A + Ckk)}$$

seu

$$y = \frac{-x\sqrt{(A + Ckk)} + k\sqrt{(A + Cxx)}}{\sqrt{A}}$$

et

$$x = \frac{-y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}}.$$

6. Si $\sqrt{(A + Cyy)}$ negative capiatur itemque $\sqrt{A}$, tum huius aequationis differentialis

$$\frac{dx}{\sqrt{(A + Cxx)}} = \frac{dy}{\sqrt{(A + Cyy)}}$$

integralis erit

$$0 = -Akk + A(xx + yy) - 2xy\sqrt{A(A + Ckk)}$$

ideoque vel

$$y = \frac{x\sqrt{(A + Ckk)} - k\sqrt{(A + Cxx)}}{\sqrt{A}}$$

vel

$$x = \frac{y\sqrt{(A + Ckk)} + k\sqrt{(A + Cyy)}}{\sqrt{A}}.$$

7. Q--ia ergo aequatio integralis constantem in se complectitur k, quae ali non inest, indicio hoc est integralem esse completam; sicque nulla alia satisfacit integralis, nisi quae in forma inventa compre- Atque haec est integratio principalis, ad quam relatio inter x et)erducit.

8. Hinc autem derivari possunt innumerabiles aliae integrationes. Si enim sint X et Y eiusmodi functiones ipsarum x et y, ut vi relationis assumtae sit $X = Y$, eadem relatio satisfaciet quoque huic aequationi differentiali

$$\frac{Xdx}{\sqrt{(A+Cxx)}} = \frac{Ydy}{\sqrt{(A+Cyy)}}.$$

Infinitis autem modis huiusmodi functiones aequales exhiberi possunt ex formulis pro x et y inventis.

9. Quo autem haec investigatio latius pateat et X et Y sint functiones similes, eas non assumo inter se aequales, eiusmodi autem pro iis valores indago, ut sit

$$\frac{Xdx}{\sqrt{(A+Cxx)}} - \frac{Ydy}{\sqrt{(A+Cyy)}} = dV$$

atque quantitas V prodeat algebraica, si scilicet relatio § 6 tradita locum habeat.

10. Cum igitur sit $\frac{dy}{\sqrt{(A+Cyy)}} = \frac{dx}{\sqrt{(A+Cxx)}}$, erit

$$\frac{(X-Y)dx}{\sqrt{(A+Cxx)}} = dV$$

et ob

$$P = k\sqrt{A(A+Cxx)} = \gamma y + \delta x = Ay + x\sqrt{A(A+Ckk)}$$

sumto per § 6 $\sqrt{A}$ negativo erit

$$\sqrt{(A+Cxx)} = \frac{x}{k}\sqrt{(A+Ckk)} - \frac{y}{k}\sqrt{A},$$

unde fiet

$$\frac{(X-Y)\,kdx}{x\sqrt{(A+Ckk)} - y\sqrt{A}} = dV.$$

11. Cum sit porro ex aequatione differentiata

$$dx(Ax - y\sqrt{A(A+Ckk)}) = dy(x\sqrt{A(A+Ckk)} - Ay),$$

ponatur $xy = u$; erit $dy = \frac{du}{x} - \frac{ydx}{x}$, quo valore substituto fiet

$$dx\left(Ax - \frac{Ayy}{x}\right) = \frac{du}{x}(x\sqrt{A(A + Ckk)} - Ay)$$

seu

$$\frac{dx}{x\sqrt{(A + Ckk)} - y\sqrt{A}} = \frac{du}{(xx - yy)\sqrt{A}}$$

sicque erit

$$dV = \frac{kdu}{\sqrt{A}} \cdot \frac{X - Y}{xx - yy}.$$

12. Quoties ergo $\frac{X - Y}{xx - yy}$ eiusmodi functio ipsius u, quae ducta in du fiat integrabilis, toties valor quantitatis V algebraice exhiberi poterit; hoc autem evenit, quoties X et Y fuerint potestates parium exponentium ipsarum x et y, propterea cum sit ex aequatione assumta

$$xx + yy = kk + \frac{2u}{A}\sqrt{A(A + Ckk)}.$$

13. Ponatur ergo $X = x^n$ et $Y = y^n$; erit posito $n = 2$

$$\frac{X - Y}{xx - yy} = 1 \quad \text{et} \quad dV = \frac{kdu}{\sqrt{A}}$$

ideoque

$$V = \frac{ku}{\sqrt{A}} + \text{Const.} = \frac{kxy}{\sqrt{A}} + \text{Const.}$$

Quam ob rem habebitur

$$\int \frac{xxdx}{\sqrt{(A + Cxx)}} - \int \frac{yydy}{\sqrt{(A + Cyy)}} = \text{Const.} + \frac{kxy}{\sqrt{A}}.$$

14. Sit iam $n = 4$ eritque

$$\frac{X - Y}{xx - yy} = xx + yy = kk + \frac{2u}{A}\sqrt{A(A + Ckk)},$$

unde

$$dV = \frac{kdu}{A}(kk\sqrt{A} + 2u\sqrt{(A + Ckk)}),$$

ergo

$$V = \frac{ku}{A}(kk\sqrt{A} + u\sqrt{(A + Ckk)}).$$

Hoc igitur casu erit

$$\int \frac{x^4 dx}{\sqrt{(A+Cxx)}} - \int \frac{y^4 dy}{\sqrt{(A+Cyy)}} = \text{Const.} + \frac{kxy}{A}(kk\sqrt{A} + xy\sqrt{(A+Ckk)})$$

similique modo ulterius progredi licet.

15. His igitur coniungendis si fuerit

$$xx + yy = kk + 2xy\sqrt{\left(1 + \frac{C}{A}kk\right)}$$

sive

$$y = \frac{x\sqrt{(A+Ckk)} - k\sqrt{(A+Cxx)}}{\sqrt{A}},$$

$$x = \frac{y\sqrt{(A+Ckk)} + k\sqrt{(A+Cyy)}}{\sqrt{A}},$$

haec relatio inter x et y satisfaciet huic aequationi integrali

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A+Cxx)}} - \int \frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{(A+Cyy)}}$$
$$= \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{\sqrt{A}}\left(kk + xy\sqrt{\left(1 + \frac{C}{A}kk\right)}\right)$$

seu differentia istarum formularum integralium algebraice assignari potest.

RELATIO SECUNDA INTER BINAS VARIABILES x ET y
$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy$

16. Quoniam, uti in praecedentibus deprehendimus, ambiguitas signorum radicalium ab arbitrio nostro pendet, dummodo eius ratio in conclusionibus finalibus debite habeatur, si ad differentiam binarum formularum integralium pervenire velimus, extrahendo radices habebimus

$$y = \frac{-\beta - \delta x - \sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta-\gamma)x + (\delta\delta - \gamma\gamma)xx)}}{\gamma},$$

$$x = \frac{-\beta - \delta y + \sqrt{(\beta\beta - \alpha\gamma + 2\beta(\delta-\gamma)y + (\delta\delta - \gamma\gamma)yy)}}{\gamma}.$$

17. Statuamus brevitatis gratia has formulas irrationales

$$\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}=P,$$
$$\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}=Q$$

eritque

$$-P=\beta+\gamma y+\delta x \quad \text{et} \quad Q=\beta+\gamma x+\delta y,$$

unde eliciuntur istae relationes

$$P+Q=(\gamma-\delta)(x-y),$$
$$\gamma P+\delta Q=\beta(\delta-\gamma)+(\delta\delta-\gamma\gamma)y,$$
$$\delta P+\gamma Q=\beta(\gamma-\delta)-(\delta\delta-\gamma\gamma)x.$$

18. Aequatio autem proposita differentiata dat

$$dx(\beta+\gamma x+\delta y)+dy(\beta+\gamma y+\delta x)=0$$

sive

$$Qdx-Pdy=0,$$

unde oritur

$$\frac{dx}{P}=\frac{dy}{Q} \quad \text{seu} \quad \int\frac{dx}{P}-\int\frac{dy}{Q}=\text{Const.},$$

cui ergo aequationi integrali satisfacit relatio proposita indeque valores pro x et y extracti.

19. Ut hinc simili modo alias integrationes obtineamus, sint iterum X et Y functiones similes ipsarum x et y ac posito

$$\frac{Xdx}{P}-\frac{Ydy}{Q}=dV$$

definiantur hae functiones ita, ut V prodeat quantitas algebraica sicque habeatur

$$\int\frac{Xdx}{P}-\int\frac{Ydy}{Q}=V+\text{Const.}$$

20. Cum igitur sit $\frac{dy}{Q}=\frac{dx}{P}$, erit

$$dV=\frac{(X-Y)dx}{P} \quad \text{seu} \quad dV=\frac{-dx(X-Y)}{\beta+\gamma y+\delta x}.$$

Sit iterum $xy = u$ ideoque $dy = \frac{du}{x} - \frac{ydx}{x}$; erit pro aequatione differentiali

$$dx(\beta + \gamma x + \delta y) + \frac{du}{x}(\beta + \gamma y + \delta x) - \frac{ydx}{x}(\beta + \gamma y + \delta x) = 0$$

seu

$$dx(\beta x - \beta y + \gamma xx - \gamma yy) + du(\beta + \gamma y + \delta x) = 0.$$

21. Valore hinc pro dx substituto habebitur

$$dV = \frac{du(X - Y)}{(x - y)(\beta + \gamma(x + y))}.$$

Ponatur autem ulterius $x + y = t$; erit $xx + yy = tt - 2u$ et aequatio assumta in hanc formam abibit

$$0 = \alpha + 2\beta t + \gamma tt + 2(\delta - \gamma)u,$$

ex qua differentiando fit

$$dt(\beta + \gamma t) = (\gamma - \delta)du$$

seu

$$\frac{du}{\beta + \gamma t} = \frac{dt}{\gamma - \delta}.$$

22. Hinc igitur simpliciori modo obtinetur

$$dV = \frac{dt(X - Y)}{(\gamma - \delta)(x - y)},$$

unde patet, si X et Y fuerint potestates ipsarum x et y, tum fractionem $\frac{X - Y}{x - y}$ per t et u ideoque et per solum t ob

$$u = \frac{\alpha + 2\beta t + \gamma tt}{2(\gamma - \delta)}$$

commode exprimi posse.

23. Sit ergo $X = x^n$ et $Y = y^n$ ac ponatur primo $n = 1$; erit $\frac{X - Y}{x - y} = 1$ et $dV = \frac{dt}{\gamma - \delta}$; unde fit $V = \frac{t}{\gamma - \delta}$. Quocirca pro hoc casu erit

$$\int \frac{xdx}{P} - \int \frac{ydy}{Q} = \text{Const.} + \frac{x + y}{\gamma - \delta},$$

cui ergo aequationi integrali satisfit per relationem inter x et y assumtam.

24. Sit $n=2$ eritque $\frac{X-Y}{x-y}=x+y=t$; unde fit

$$dV=\frac{tdt}{\gamma-\delta} \quad \text{et} \quad V=\frac{tt}{2(\gamma-\delta)}=\frac{(x+y)^2}{2(\gamma-\delta)}.$$

Hoc ergo casu habebitur

$$\int\frac{xxdx}{P}-\int\frac{yydy}{Q}=\text{Const.}+\frac{(x+y)^2}{2(\gamma-\delta)}.$$

25. Si ulterius progredi lubeat, ponatur $n=3$ eritque

$$\frac{x^3-y^3}{x-y}=xx+xy+yy=tt-u=\frac{(\gamma-2\delta)tt-2\beta t-\alpha}{2(\gamma-\delta)}$$

et

$$V=\frac{\frac{1}{3}(\gamma-2\delta)t^3-\beta tt-\alpha t}{2(\gamma-\delta)^2}$$

sicque erit

$$\int\frac{x^3dx}{P}-\int\frac{y^3dy}{Q}=\text{Const.}+\frac{(\gamma-2\delta)(x+y)^3-3\beta(x+y)^2-3\alpha(x+y)}{6(\gamma-\delta)^2}.$$

26. His igitur formulis coniungendis sequenti aequationi integrali

$$\int\frac{dx(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx+\mathfrak{D}x^3)}{\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)x+(\delta\delta-\gamma\gamma)xx)}}-\int\frac{dy(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}yy+\mathfrak{D}y^3)}{\sqrt{(\beta\beta-\alpha\gamma+2\beta(\delta-\gamma)y+(\delta\delta-\gamma\gamma)yy)}}$$
$$=\text{Const.}+\frac{\mathfrak{B}(x+y)}{\gamma-\delta}+\frac{\mathfrak{C}(x+y)^2}{2(\gamma-\delta)}+\frac{\mathfrak{D}((\gamma-2\delta)(x+y)^3-3\beta(x+y)^2-3\alpha(x+y))}{6(\gamma-\delta)^2}$$

satisfacit relatio assumta

$$0=\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy$$

indeque valores pro x et y initio eruti.

27. Quo applicatio ad casus particulares facilius fieri possit, ponamus

$$\beta\beta-\alpha\gamma=Ap, \quad \beta(\delta-\gamma)=Bp \quad \text{et} \quad \delta\delta-\gamma\gamma=Cp,$$

ut sit

$$P=\sqrt{p(A+2Bx+Cxx)} \quad \text{et} \quad Q=\sqrt{p(A+2By+Cyy)}$$

fiatque

$$\gamma=A+Bk \quad \text{et} \quad \delta=\sqrt{A(A+2Bk+Ckk)};$$

erit

$$p = \frac{(AC-BB)kk}{C} \quad \text{et} \quad \beta = \frac{B}{C}(\delta + \gamma)$$

atque

$$\alpha = \frac{2BB}{CC}(\gamma + \delta) - \frac{(AC-BB)^2 kk}{CC(A+Bk)}.$$

RELATIO TERTIA INTER BINAS VARIABILES x ET y

$$0 = \alpha + mxx + nyy + 2\delta xy$$

28. Extrahendo utramque radicem habebitur

$$y = \frac{-\delta x + \sqrt{((\delta\delta - mn)xx - \alpha n)}}{n},$$

$$x = \frac{-\delta y - \sqrt{((\delta\delta - mn)yy - \alpha m)}}{m};$$

hinc posito

$$P = \sqrt{((\delta\delta - mn)xx - \alpha n)} \quad \text{et} \quad Q = \sqrt{((\delta\delta - mn)yy - \alpha m)}$$

erit

$$P = \delta x + ny \quad \text{et} \quad -Q = \delta y + mx.$$

29. Per differentiationem vero obtinemus

$$dx(mx + \delta y) + dy(ny + \delta x) = 0$$

seu $-Qdx + Pdy = 0$ ideoque $\frac{dy}{Q} = \frac{dx}{P}$, unde aequatio assumta huic aequationi integrali

$$\int \frac{dy}{Q} = \int \frac{dx}{P}$$

satisfacit.

30. Sint iam X et Y functiones ipsarum x et y singulatim ac ponatur

$$\int \frac{Xdx}{P} - \int \frac{Ydy}{Q} = V,$$

ita ut fiat V quantitas algebraica, eritque

$$\frac{(X-Y)dx}{P} = dV = \frac{(X-Y)dx}{\delta x + ny}.$$

31. Posito $xy = u$, ut sit $dy = \frac{du}{x} - \frac{y\,dx}{x}$, erit

$$dx(mxx - nyy) + du(ny + \delta x) = 0,$$

unde, cum fiat $\frac{dx}{\delta x + ny} = \frac{-du}{mxx - nyy}$, erit

$$dV = \frac{-du(X - Y)}{mxx - nyy}$$

hincque non difficulter casus integrabiles eliciuntur.

32. Sit enim primo $X = mxx$ et $Y = nyy$; erit

$$dV = -du \quad \text{et} \quad V = -u = -xy.$$

Hinc relatio inter x et y assumta satisfacit huic aequationi integrali

$$\int \frac{mxx\,dx}{P} - \int \frac{nyy\,dy}{Q} = \text{Const.} - xy.$$

33. Sit secundo $X = mmx^4$ et $Y = nny^4$; erit

$$dV = -du(mxx + nyy) = +du(\alpha + 2\delta u),$$

unde fit

$$V = u(\alpha + \delta u) = xy(\alpha + \delta xy).$$

Ergo huic aequationi integrali

$$\int \frac{mmx^4 dx}{P} - \int \frac{nny^4 dy}{Q} = \text{Const.} + xy(\alpha + \delta xy)$$

satisfacit relatio assumta inter x et y.

34. His igitur colligendis relatio inter x et y assumta satisfaciet huic aequationi integrali

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}mxx + \mathfrak{C}m^2x^4)}{\sqrt{((\delta\delta - mn)xx - \alpha n)}} - \int \frac{dy(\mathfrak{A} + \mathfrak{B}nyy + \mathfrak{C}n^2y^4)}{\sqrt{((\delta\delta - mn)yy - \alpha m)}}$$
$$= \text{Const.} - \mathfrak{B}xy + \mathfrak{C}xy(\alpha + \delta xy).$$

35. Ponamus ad faciliorem applicationem

$$\delta\delta - mn = Cp, \quad \alpha n = -Ap \quad \text{et} \quad \alpha m = -Bp,$$

ut sit

$$P = \sqrt{p(A + Cxx)} \quad \text{et} \quad Q = \sqrt{p(B + Cyy)};$$

erit $\frac{m}{n} = \frac{B}{A}$. Sit ergo $m = B$ et $n = A$; erit

$$\alpha = -p \quad \text{et} \quad \delta = \sqrt{(AB + Cp)}.$$

Sit ergo $p = Ckk$, ut sit $\alpha = -Ckk$, et aequatio relationem inter x et y definiens erit

$$0 = -Ckk + Bxx + Ayy + 2xy\sqrt{(AB + CCkk)}.$$

36. Quam ob rem valores ipsius x et y hinc erunt

$$y = \frac{-x\sqrt{(AB + CCkk)} + k\sqrt{C(A + Cxx)}}{A},$$

$$x = \frac{-y\sqrt{(AB + CCkk)} - k\sqrt{C(B + Cyy)}}{B}$$

existente

$$P = k\sqrt{C(A + Cxx)} \quad \text{et} \quad Q = k\sqrt{C(B + Cyy)}.$$

37. Hi igitur valores conveniunt huic aequationi integrali

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}Bxx + \mathfrak{C}B^2x^4)}{\sqrt{(A + Cxx)}} - \int \frac{dy(\mathfrak{A} + \mathfrak{B}Ayy + \mathfrak{C}A^2y^4)}{\sqrt{(B + Cyy)}}$$
$$= \text{Const.} - \mathfrak{B}kxy\sqrt{C} + \mathfrak{C}kxy(-Ckk + xy\sqrt{(AB + C^2kk)})\sqrt{C}.$$

38. Ponatur $B = \frac{CE}{F}$, quae aequatio latius patere videatur, atque constantibus mutatis prodibit ista aequatio integralis

$$\int \frac{dx\left(\mathfrak{A} + \frac{C}{A}\mathfrak{B}xx + \frac{CC}{AA}\mathfrak{C}x^4\right)\sqrt{C}}{\sqrt{(A + Cxx)}} - \int \frac{dy\left(\mathfrak{A} + \frac{F}{E}\mathfrak{B}yy + \frac{FF}{EE}\mathfrak{C}y^4\right)\sqrt{F}}{\sqrt{(E + Fyy)}}$$
$$= \text{Const.} - \frac{CF}{AE}\mathfrak{B}kxy - \frac{CCFF}{AAEE}\mathfrak{C}k^3xy + \frac{CCFF}{AAEE}\mathfrak{C}kxxyy\sqrt{\left(\frac{AE}{CF} + kk\right)},$$

cui satisfaciunt isti valores

$$\frac{Ay}{C} = k\sqrt{\left(\frac{A}{C}+xx\right)} - x\sqrt{\left(\frac{AE}{CF}+kk\right)},$$

$$\frac{Ex}{F} = -k\sqrt{\left(\frac{E}{F}+yy\right)} - y\sqrt{\left(\frac{AE}{CF}+kk\right)},$$

qui oriuntur ex hac aequatione

$$kk = \frac{E}{F}xx + \frac{A}{C}yy + 2xy\sqrt{\left(\frac{AE}{CF}+kk\right)}.$$

39. Hae formulae ratione signorum utcunque transmutari possunt. Primo enim in formulis integralibus nihil mutando tam k quam $\sqrt{\left(\frac{AE}{CF}+kk\right)}$ pro lubitu vel affirmative vel negative accipi possunt, dummodo eadem signi ratio ubique observetur. Deinde etiam tam $\sqrt{C}$ quam $\sqrt{F}$ negative sumi potest; illo autem casu quoque $\sqrt{\left(\frac{A}{C}+xx\right)}$, quippe $\frac{\sqrt{(A+Cxx)}}{\sqrt{C}}$, hoc vero $\sqrt{\left(\frac{E}{F}+yy\right)}$ negative est accipiendum.

40. Denique patet, si C sit quantitas positiva, tum quoque F quantitatem positivam esse oportere, quia alioquin altera formula integralis fieret imaginaria. Sin autem C sit quantitas negativa, tum etiam F talis sit necesse est; et quo hoc casu imaginaria se destruant, pro kk quantitas negativa accipienda erit, quo k et k^3 fiant quoque imaginariae.

41. Hoc ergo casu sequens habebitur aequatio integralis

$$\int \frac{dx\left(\mathfrak{A}+\frac{C}{A}\mathfrak{B}xx+\frac{CC}{AA}\mathfrak{C}x^4\right)\sqrt{C}}{\sqrt{(A-Cxx)}} - \int \frac{dy\left(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy+\frac{FF}{EE}\mathfrak{C}y^4\right)\sqrt{F}}{\sqrt{(E-Fyy)}}$$

$$= \text{Const.} + \frac{CF}{AE}\mathfrak{B}kxy + \frac{CCFF}{AAEE}\mathfrak{C}k^3xy + \frac{CCFF}{AAEE}\mathfrak{C}kxxyy\sqrt{\left(\frac{AE}{CF}-kk\right)},$$

cui satisfaciunt isti valores

$$\frac{Ay}{C} = x\sqrt{\left(\frac{AE}{CF}-kk\right)} - k\sqrt{\left(\frac{A}{C}-xx\right)},$$

$$\frac{Ex}{F} = y\sqrt{\left(\frac{AE}{CF}-kk\right)} + k\sqrt{\left(\frac{E}{F}-yy\right)}$$

ex hac aequatione oriundi

$$kk = \frac{E}{F}xx + \frac{A}{C}yy - 2xy\sqrt{\left(\frac{AE}{CF} - kk\right)}.$$

42. Hae formulae etiam eas, quae ex hypothesi prima sunt erutae, in se complectuntur, ponendo scilicet $E = A$ et $F = C$; quin etiam formulae secundae hypothesis his non latius patent. Si enim in relatione secundo loco assumta pro $x + \frac{\beta}{\gamma+\delta}$ et $y + \frac{\beta}{\gamma+\delta}$ scribatur x et y, aequatio omnino primae formae oritur similique modo, si hanc relationem constituere velimus

$$0 = \alpha + 2bx + 2\beta y + \gamma xx + cyy + 2\delta xy,$$

ea facile ad relationem tertiam reduceretur, unde eius evolutionem praetermitto.

43. Perspicuum nunc est ex his formulis infinitas comparationes institui posse circa quantitates transcendentes tam ratione spatiorum quam arcuum, qui quidem vel a quadratura circuli pendent vel a logarithmis. Etsi autem hae comparationes etiam vulgari calculo institui possunt, tamen non inutile erit ostendere, quemadmodum eaedem multo facilius ex his formulis derivari queant; quod eo magis notatu dignum videtur, cum hic neque naturae circuli neque logarithmorum ratio peculiaris habeatur. Ex quo facilius intelligetur, quemadmodum haec methodus etiam pari successu ad eiusmodi formulas integrales se extendat, quae neque ad circuli neque hyperbolae quadraturam revocari possunt.

I. DE COMPARATIONE ARCUUM CIRCULARIUM

44. Sit radius circuli seu sinus totus $= 1$ ac posito sinu quocunque $= z$ sit arcus ei respondens $= \Pi . z$, sumto Π pro nota eius functionis, qua pendentia arcus a suo sinu denotatur. Erit ergo, uti constat,

$$\Pi . z = \int \frac{dz}{\sqrt{(1 - zz)}};$$

atque ut formulas integrales § 41 erutas huc transferamus, poni oportet

$$A = E = C = F = 1, \quad \mathfrak{A} = 1, \quad \mathfrak{B} = 0 \quad \text{et} \quad \mathfrak{C} = 0.$$

45. Ex his autem valoribus emerget haec aequatio integralia complectens

$$\int\frac{dx}{\sqrt{(1-xx)}}-\int\frac{dy}{\sqrt{(1-yy)}}=\text{Const.},$$

cui satisfacere inventae sunt hae formulae

$$y=x\sqrt{(1-kk)}-k\sqrt{(1-xx)},$$

$$x=y\sqrt{(1-kk)}+k\sqrt{(1-yy)},$$

quae oriuntur ex hac aequatione

$$kk=xx+yy-2xy\sqrt{(1-kk)}.$$

46. Per has igitur determinationes satisfit huic aequationi

$$\Pi.x-\Pi.y=\text{Const.},$$

in qua constans ita determinabitur: ponatur $y=0$ eritque $x=k$; ex quo casu prodit $\Pi.k-\Pi.0=\text{Const.}$ seu ob $\Pi.0=0$ erit $\text{Const.}=\Pi.k$ seu arcui, cuius sinus $=k$. Hinc generatim habebimus

$$\Pi.x-\Pi.y=\Pi.k.$$

47. Hinc ergo statim arcuum tam additio quam subtractio colligitur. Si enim duo habeantur arcus $\Pi.k$ et $\Pi.y$, quarum sinus sint k et y, et summae arcuum sinus ponatur $=x$, ut sit $\Pi.x=\Pi.k+\Pi.y$, erit

$$x=y\sqrt{(1-kk)}+k\sqrt{(1-yy)}.$$

Porro si maioris arcus sinus sit $=x$, minoris $=k$ sinusque differentiae ponatur $=y$, ut sit $\Pi.y=\Pi.x-\Pi.k$, erit

$$y=x\sqrt{(1-kk)}-k\sqrt{(1-xx)},$$

uti ex elementis est manifestum.

48. Perspicuum etiam est, quemadmodum hinc arcuum multiplicationem deduci oporteat. Posito enim $y=k$, ut sit

$$x=2k\sqrt{(1-kk)},$$

erit

$$\Pi.x=2\Pi.k.$$

Ac si valor hic pro x inventus loco y substituatur, in formula

$$x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)}$$

ob $\Pi. y = 2\Pi. k$ prodibit

$$\Pi. x = 3\Pi. k.$$

49. In genere autem, si sit y sinus arcus nk seu $\Pi. y = n\Pi. k$ et $\sqrt{(1-yy)}$ sit cosinus arcus nk, uti $\sqrt{(1-kk)}$ denotat cosinum arcus k, atque ponatur $x = y\sqrt{(1-kk)} + k\sqrt{(1-yy)}$, erit

$$\Pi. x = (n+1)\Pi. k.$$

Ex sinu ergo cuiusvis multipli arcus k reperietur sinus multipli unitate altioris.

50. Quo autem haec facilius expediri queant, valorem quoque ipsius cosinus $\sqrt{(1-xx)}$ nosse conveniet; quem in finem, cum ex formula prima sit

$$k\sqrt{(1-xx)} = x\sqrt{(1-kk)} - y,$$

substituatur hic valor ipsius x ex altera formula; erit

$$k\sqrt{(1-xx)} = y(1-kk) + k\sqrt{(1-kk)(1-yy)} - y$$

ideoque

$$\sqrt{(1-xx)} = \sqrt{(1-kk)(1-yy)} - ky$$

similique modo erit

$$\sqrt{(1-yy)} = \sqrt{(1-kk)(1-xx)} + kx.$$

51. Inventis ergo valoribus tam pro x quam pro $\sqrt{(1-xx)}$ multiplicetur ille per λ et productum ad hunc addatur eritque

$$\sqrt{(1-xx)} + \lambda x = \sqrt{(1-kk)(1-yy)} - ky + \lambda y\sqrt{(1-kk)} + \lambda k\sqrt{(1-yy)}$$

seu

$$\sqrt{(1-xx)} + \lambda x = \left(\sqrt{(1-kk)} + \lambda k\right)\sqrt{(1-yy)} + y\left(\lambda\sqrt{(1-kk)} - k\right).$$

Quo igitur hi factores similes reddantur, necesse est, ut sit $\lambda = \sqrt{-1}$, eritque

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right)\left(\sqrt{(1-yy)} + y\sqrt{-1}\right).$$

52. Hanc ergo formulam loco superioris adhibendo statim patet, ut sit $\Pi. x = 2\Pi. k$, ob $y = k$ esse oportere

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^2.$$

Ac si hic valor pro x inventus loco y substituatur, ut sit $\Pi. y = 2\Pi. k$, prodibit

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^3$$

pro $\Pi. x = 3\Pi. k$, unde in genere colligitur, ut sit $\Pi. x = n\Pi. k$, debere esse

$$\sqrt{(1-xx)} + x\sqrt{-1} = \left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^n.$$

53. Quia porro $\sqrt{-1}$ ob suam naturam tam negative quam affirmative accipere licet, erit quoque pro eadem arcus multiplicatione $\Pi. x = n\Pi. k$

$$\sqrt{(1-xx)} - x\sqrt{-1} = \left(\sqrt{(1-kk)} - k\sqrt{-1}\right)^n$$

ideoque vel

$$\sqrt{(1-xx)} = \frac{\left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^n + \left(\sqrt{(1-kk)} - k\sqrt{-1}\right)^n}{2}$$

vel

$$x = \frac{\left(\sqrt{(1-kk)} + k\sqrt{-1}\right)^n - \left(\sqrt{(1-kk)} - k\sqrt{-1}\right)^n}{2\sqrt{-1}},$$

quae formulae quoque valent pro valoribus fractis exponentis n.

II. DE COMPARATIONE ARCUUM PARABOLICORUM

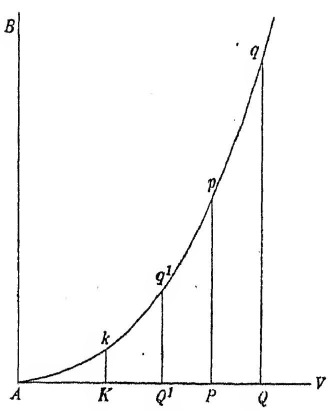

Fig. 1.

54. Sit AB (Fig. 1) axis et A vertex parabolae, quem tangat recta indefinita AV, super qua capiantur abscissae; posito ergo parabolae latere recto $= 2$ sit abscissa quaevis $AP = z$; erit applicata $Pp = \frac{1}{2}zz$, ex quo arcus parabolae huic abscissae respondens erit $Ap = \int dz\sqrt{(1+zz)}$; qui cum sit functio ipsius z, denotetur per $\Pi. z$, ita ut $\Pi. z$ significet arcum parabolae abscissae z convenientem seu sit

$$\Pi. z = \int dz\sqrt{(1+zz)}.$$

55. Irrationalitate in denominatorem translata erit

$$\Pi. z = \int \frac{dz(1+zz)}{\sqrt{(1+zz)}}.$$

Ad hanc ergo formam ut formulae integrales § 38 revocentur, erit

$$A = E = 1, \quad C = F = 1, \quad \mathfrak{A} = 1 \quad \text{et} \quad \mathfrak{B} = 1 \quad \text{atque} \quad \mathfrak{C} = 0.$$

Unde aequatio illa integralis in hanc abit formam

$$\int \frac{dx(1+xx)}{\sqrt{(1+xx)}} - \int \frac{dy(1+yy)}{\sqrt{(1+yy)}} = \text{Const.} + kxy,$$

cui satisfaciunt hi valores

$$y = -k\sqrt{(1+xx)} + x\sqrt{(1+kk)} \quad \text{et} \quad x = k\sqrt{(1+yy)} + y\sqrt{(1+kk)}$$

sumtis tam k quam $\sqrt{(1+kk)}$ negativis.

56. Hac igitur inter x et y relatione subsistente pro arcubus parabolae erit

$$\Pi. x - \Pi. y = \text{Const.} + kxy;$$

ad quam constantem determinandam ponatur $y = 0$, et quia tum fit $x = k$, erit $\Pi. k = \text{Const.}$ Quocirca habebitur

$$\Pi. x - \Pi. y = \Pi. k + kxy.$$

57. Ut igitur haec aequatio locum habeat, relatio inter ternas abscissas k, x et y eiusmodi erit

$$x = k\sqrt{(1+yy)} + y\sqrt{(1+kk)} \quad \text{seu} \quad y = x\sqrt{(1+kk)} - k\sqrt{(1+xx)},$$

unde praeterea eruuntur istae determinationes

$$\sqrt{(1+xx)} = \sqrt{(1+kk)(1+yy)} + ky \quad \text{et} \quad \sqrt{(1+yy)} = \sqrt{(1+kk)(1+xx)} - kx,$$

ex quibus porro elicitur

$$x + \sqrt{(1+xx)} = (k + \sqrt{(1+kk)})(y + \sqrt{(1+yy)}).$$

58. Si manente eadem abscissa k capiantur aliae duae abscissae q et p, ut sit

$$q = k\sqrt{(1+pp)} + p\sqrt{(1+kk)} \quad \text{et} \quad p = q\sqrt{(1+kk)} - k\sqrt{(1+qq)}$$

seu

$$q + \sqrt{(1+qq)} = (k + \sqrt{(1+kk)})(p + \sqrt{(1+pp)}),$$

erit

$$\Pi.\, q - \Pi.\, p = \Pi.\, k + kpq.$$

Ideoque hanc aequationem ab illa subtrahendo habebitur

$$(\Pi.\, x - \Pi.\, y) - (\Pi.\, q - \Pi.\, p) = k(xy - pq).$$

59. Pro hoc igitur casu erit

$$\frac{x + \sqrt{(1+xx)}}{y + \sqrt{(1+yy)}} = \frac{q + \sqrt{(1+qq)}}{p + \sqrt{(1+pp)}},$$

unde relatio inter p, q, x et y sine k obtinetur. Erit autem

$$k = x\sqrt{(1+yy)} - y\sqrt{(1+xx)} = q\sqrt{(1+pp)} - p\sqrt{(1+qq)}$$

et

$$\sqrt{(1+kk)} = \sqrt{(1+xx)(1+yy)} - xy = \sqrt{(1+pp)(1+qq)} - pq.$$

60. Iam ob

$$\frac{1}{p + \sqrt{(1+pp)}} = \sqrt{(1+pp)} - p$$

erit

$$\sqrt{(1+xx)} + x = (\sqrt{(1+yy)} + y)(\sqrt{(1+qq)} + q)(\sqrt{(1+pp)} - p),$$

unde reperitur

$$x = y\sqrt{(1+pp)(1+qq)} + q\sqrt{(1+pp)(1+yy)} - p\sqrt{(1+qq)(1+yy)} - pqy.$$

Quare erit

$$(\Pi.\, x - \Pi.\, y) - (\Pi.\, q - \Pi.\, p)$$
$$= (q\sqrt{(1+pp)} - p\sqrt{(1+qq)})(y\sqrt{(1+pp)} - p\sqrt{(1+yy)})(q\sqrt{(1+yy)} + y\sqrt{(1+qq)}).$$

PROBLEMA 1

61. *Dato arcu parabolae quocunque Ak* (Fig. 1, p. 128) *in vertice A terminato ab alio quocunque puncto p arcum abscindere pq, qui arcum illum Ak superet quantitate algebraice assignabili.*

SOLUTIO

Posita parabolae parametro $= 2$ sit k abscissa arcui Ak conveniens, abscissae autem punctis p et q respondentes sint $AP = y$ et $AQ = x$ eritque

$$\text{Arc.}\, pq = \Pi.\, x - \Pi.\, y \quad \text{et} \quad \text{Arc.}\, Ak = \Pi.\, k;$$

cum igitur data sit abscissa $AP = y$, si capiatur altera

$$AQ = x = y\sqrt{(1 + kk)} + k\sqrt{(1 + yy)},$$

erit

$$\Pi.\, x - \Pi.\, y = \Pi.\, k + kxy$$

ideoque

$$\text{Arc.}\, pq = \text{Arc.}\, Ak + kxy.$$

Superabit ergo arcus pq, qui in dato puncto p terminatur, arcum Ak quantitate algebraice assignabili kxy.

Poterit etiam a puncto p antrorsum abscindi arcus pq^1, qui pariter arcum Ak quantitate geometrica superet; ad hoc ponatur $AP = x$ et $AQ^1 = y$ sitque $y = x\sqrt{(1 + kk)} - k\sqrt{(1 + xx)}$; et cum sit $\text{Arc.}\, pq^1 = \Pi.\, x - \Pi.\, y$, erit

$$\text{Arc.}\, pq^1 = \text{Arc.}\, Ak + kxy.$$

Utraque igitur solutio ita coniungetur, ut posita abscissa data $AP = p$ capiendum sit

$$AQ = p\sqrt{(1 + kk)} + k\sqrt{(1 + pp)} \quad \text{et} \quad AQ^1 = p\sqrt{(1 + kk)} - k\sqrt{(1 + pp)},$$

quo facto erit

$$\text{Arc.}\, pq = \text{Arc.}\, Ak + kp \cdot AQ,$$
$$\text{Arc.}\, pq^1 = \text{Arc.}\, Ak + kp \cdot AQ^1,$$

sicque duplici modo problemati est satisfactum.

COROLLARIUM 1

62. Fieri autem nequit, ut excessus kxy, quo arcus pq arcum Ak superat, evanescat; deberet enim esse vel $x = 0$ vel $y = 0$. At casu $x = 0$

fieret $y = -k$ arcusque in ipso vertice A inciperet in altero ramo ipsi arcui Ak similis capiendus; altero autem casu, quo $y = 0$, fieret $x = k$ et arcus pq in arcum Ak abiret; unde arcui Ak geometrice in parabola abscindi nequit alius arcus ipsi aequalis, qui ipsi non simul futurus sit similis.

COROLLARIUM 2

63. Vicissim ergo dato arcu quocunque pq in parabola semper a vertice arcus abscindi poterit Ak, qui ab illo deficiat quantitate geometrica. Cum enim nunc datae sint abscissae $AP = y$ et $AQ = x$, erit

$$AK = k = x\sqrt{(1+yy)} - y\sqrt{(1+xx)},$$

qua inventa erit Arc. pq — Arc. $Ak = kxy$.

COROLLARIUM 3

64. Quin etiam puncto p pro incognito habito, proposito arcu Ak, alius arcus pq assignari poterit, qui illum superet quantitate data, puta $= C$. Habebimus ergo has duas aequationes

$$kxy = C \quad \text{et} \quad xx + yy = kk + 2xy\sqrt{(1+kk)}$$

seu

$$xx + yy = kk + \frac{2C}{k}\sqrt{(1+kk)};$$

ergo

$$x + y = \sqrt{\left(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)}\right)},$$

$$x - y = \sqrt{\left(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)}\right)}.$$

Seu sint x et y binae radices huius aequationis quadraticae

$$zz - Pz + Q = 0;$$

erit

$$Q = \frac{C}{k} \quad \text{et} \quad P = \sqrt{\left(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)}\right)},$$

unde

$$z = \frac{1}{2}\sqrt{\left(kk + \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)}\right)} \pm \frac{1}{2}\sqrt{\left(kk - \frac{2C}{k} + \frac{2C}{k}\sqrt{(1+kk)}\right)}.$$

COROLLARIUM 4

65. Quantacunque sit haec quantitas C, modo sit affirmativa, semper prodeunt pro x et y valores reales iique affirmativi. At si sit $C=0$, fiet $x=k$ et $y=0$. Quin etiam poni potest C negativum, quo casu y reperitur quoque negativum et arcus quaesitus utrinque circa verticem A erit dispositus. Verum si sit $C=-D$, necesse est, ut sit

$$D<\frac{k^3}{2(1+\sqrt{(1+kk)})} \quad \text{seu} \quad D<\frac{1}{2}k\left(\sqrt{(1+kk)}-1\right);$$

nam si D esset maius, utraque abscissa fieret imaginaria.

COROLLARIUM 5

66. Casu autem

$$D=-C=\frac{1}{2}k\left(\sqrt{(1+kk)}-1\right) \quad \text{erit} \quad zz=\frac{D}{k}$$

ideoque

$$x=+\sqrt{\tfrac{1}{2}\left(\sqrt{(1+kk)}-1\right)} \quad \text{et} \quad y=-\sqrt{\tfrac{1}{2}\left(\sqrt{(1+kk)}-1\right)};$$

hocque casu orietur arcus utrinque a vertice aeque extensus, cuius defectus ab arcu Ak est minimus omnium, qui quidem geometrice construi possunt.

PROBLEMA 2

67. *Dato arcu parabolae quocunque ef* (Fig. 2) *a dato eius puncto quocunque p alium abscindere arcum pq, ita ut arcuum ef et pq differentia geometrice possit assignari.*

SOLUTIO

Posito parabolae latere recto $=2$ tanget recta AV parabolam in vertice A, a quo capiantur abscissae, quae sint $AE=e$, $AF=f$, $AP=p$ et $AQ=q$, quarum tres priores e, f, p sunt datae, haec vera q ita accipiatur, ut sit per § 59

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}}=\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}.$$

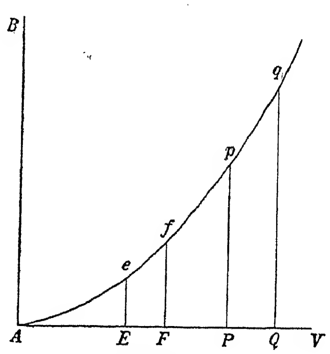

Fig. 2.

Tum vero fit

$$k = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$$

scribendo e et f pro y et x eritque

$$(\Pi.\, q - \Pi.\, p) - (\Pi.\, f - \Pi.\, e) = k(pq - ef).$$

Ideoque habebitur

$$\text{Arc.}\, pq - \text{Arc.}\, ef = k(pq - ef).$$

Hinc etiam apparet, si punctum q fuerit datum, ex formula tradita simili modo punctum p antrorsum procedendo definiri posse, ut arcuum differentia prodeat geometrice assignabilis.

COROLLARIUM 1

68. Ex reductione § 60 facta patet esse

$$pq - ef = \left(p\sqrt{(1+ee)} - e\sqrt{(1+pp)}\right)\left(p\sqrt{(1+ff)} + f\sqrt{(1+pp)}\right)$$

sicque sumta abscissa q ex aequatione

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}$$

erit

$$\text{Arc.}\, pq - \text{Arc.}\, ef$$
$$= \left(f\sqrt{(1+ee)} - e\sqrt{(1+ff)}\right)\left(p\sqrt{(1+ee)} - e\sqrt{(1+pp)}\right)\left(p\sqrt{(1+ff)} + f\sqrt{(1+pp)}\right).$$

COROLLARIUM 2

69. Si velimus punctum p ita accipere, ut arcuum differentia evanescat seu fiat $\text{Arc.}\, pq = \text{Arc.}\, ef$, oportet esse

$$\text{vel } p\sqrt{(1+ee)} - e\sqrt{(1+pp)} = 0 \quad \text{vel } p\sqrt{(1+ff)} + f\sqrt{(1+pp)} = 0.$$

Priori casu fit $p = \pm e$, posteriori $p = \pm f$, utroque autem casu arcus pq vel cum arcu ef congruit vel eius fit similis in altero parabolae ramo assumtus, ita ut geometrice duo arcus aequales exhiberi nequeant, quae non simul sibi futuri sint similes.

COROLLARIUM 3

70. Cum sit $k = f\sqrt{(1+ee)} - e\sqrt{(1+ff)}$, erit

$$\sqrt{(1+kk)} = \sqrt{(1+ee)(1+ff)} - ef;$$

hinc

$$k\sqrt{(1+kk)} = f\sqrt{(1+ff)} + 2eef\sqrt{(1+ff)} - 2eff\sqrt{(1+ee)} - e\sqrt{(1+ee)}$$

sive

$$k\sqrt{(1+kk)} = f\sqrt{(1+ff)} - e\sqrt{(1+ee)} - 2ef\left(f\sqrt{(1+ee)} - e\sqrt{(1+ff)}\right)$$

ideoque

$$k\sqrt{(1+kk)} = f\sqrt{(1+ff)} - e\sqrt{(1+ee)} - 2efk.$$

Quo circa habebitur

$$kef = \frac{1}{2}f\sqrt{(1+ff)} - \frac{1}{2}e\sqrt{(1+ee)} - \frac{1}{2}k\sqrt{(1+kk)}.$$

COROLLARIUM 4

71. Quia igitur k simili quoque modo pendet a p et q, erit etiam

$$kpq = \frac{1}{2}q\sqrt{(1+qq)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{1}{2}k\sqrt{(1+kk)}.$$

Quare cum arcuum differentia sit $= kpq - kef$, si quatuor parabolae puncta e, f, p, q ita a se invicem pendent, ut sit

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}},$$

erit

$$\text{Arc.}\, pq - \text{Arc.}\, ef = \frac{1}{2}q\sqrt{(1+qq)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{1}{2}f\sqrt{(1+ff)} + \frac{1}{2}e\sqrt{(1+ee)},$$

quae expressio ob functiones quantitatum p, q, e, f a se invicem separatas est notatu digna.

COROLLARIUM 5

72. Relatio inter e, f, p, q etiam ita exprimi potest, ut sit

$$\sqrt{(1+qq)} + q = \left(\sqrt{(1+ee)} - e\right)\left(\sqrt{(1+ff)} + f\right)\left(\sqrt{(1+pp)} + p\right);$$

tum ob

$$\frac{1}{\sqrt{(1+qq)}+q} = \sqrt{(1+qq)} - q$$

erit

$$\sqrt{(1+qq)} - q = \left(\sqrt{(1+ee)}+e\right)\left(\sqrt{(1+ff)}-f\right)\left(\sqrt{(1+pp)}-p\right),$$

unde datis e, f et p facile valor tam pro q quam pro $\sqrt{(1+qq)}$ eruitur.

COROLLARIUM 6

73. Ex formula corollario 1 data apparet arcum pq semper maiorem fore arcu ef, si punctum p a vertice parabolae A magis fuerit remotum quam punctum e, contra autem arcum pq proditurum esse minorem. Ac si quidem sit $p = 0$, erit

$$\text{Arc. } ef - \text{Arc. } pq = ef\left(f\sqrt{(1+ee)} - e\sqrt{(1+ff)}\right);$$

minimus autem omnium arcus pq evadet, si capiatur

$$p = -\sqrt{\tfrac{1}{2}\left(\sqrt{(1+ee)(1+ff)} - ef - 1\right)}$$

et

$$q = +\sqrt{\tfrac{1}{2}\left(\sqrt{(1+ee)(1+ff)} - ef - 1\right)},$$

tumque erit

$$\text{Arc. } ef - \text{Arc. } pq = \frac{1}{2}(e+f)\left(\sqrt{(1+ff)} - \sqrt{(1+ee)}\right).$$

Arcusque pq utrinque aeque circa verticem A erit dispositus.

PROBLEMA 3

74. *Dato arcu parabolae ef* (Fig. 3, p. 137) *a puncto dato p abscindere arcum pz, qui superet datum multiplum arcus ef quantitate geometrice assignabili.*

SOLUTIO

Posito parabolae latere recto $= 2$ sint in verticis tangente abscissae datae $AE = e$, $AF = f$ et $AP = p$; tum capiantur abscissae $AQ = q$, $AR = r$,

$AS = s$, $AT = t$ et ultima sit $AZ = z$; quae ita determinentur, ut sit primo

$$\frac{q+\sqrt{(1+qq)}}{p+\sqrt{(1+pp)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}},$$

eritque ex § 71

$$\text{Arc.}\,pq - \text{Arc.}\,ef = \frac{1}{2}q\sqrt{(1+qq)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{1}{2}f\sqrt{(1+ff)} + \frac{1}{2}e\sqrt{(1+ee)}.$$

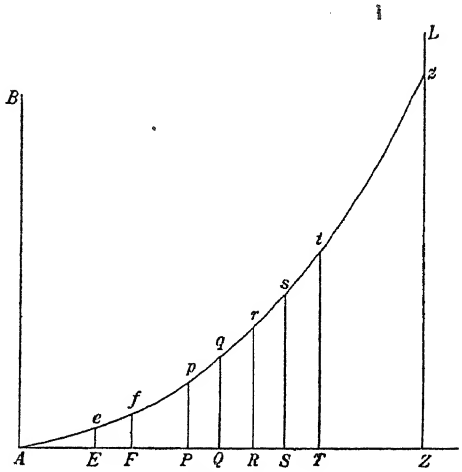

Fig. 3.

Deinde ex puncto q simili modo definiatur punctum r, ut sit

$$\frac{r+\sqrt{(1+rr)}}{q+\sqrt{(1+qq)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}} \quad \text{seu} \quad \frac{r+\sqrt{(1+rr)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^2,$$

eritque

$$\text{Arc.}\,qr - \text{Arc.}\,ef = \frac{1}{2}r\sqrt{(1+rr)} - \frac{1}{2}q\sqrt{(1+qq)} - \frac{1}{2}f\sqrt{(1+ff)} + \frac{1}{2}e\sqrt{(1+ee)},$$

qua aequatione ad illam addita prodibit

$$\text{Arc.}\,pr - 2\,\text{Arc.}\,ef = \frac{1}{2}r\sqrt{(1+rr)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{2}{2}f\sqrt{(1+ff)} + \frac{2}{2}e\sqrt{(1+ee)}.$$

Tertio ex puncto r capiatur punctum s, ut sit

$$\frac{s+\sqrt{(1+ss)}}{r+\sqrt{(1+rr)}} = \frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}} \quad \text{seu} \quad \frac{s+\sqrt{(1+ss)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^3,$$

eritque

$$\text{Arc.}\,rs - \text{Arc.}\,ef = \frac{1}{2}s\sqrt{(1+ss)} - \frac{1}{2}r\sqrt{(1+rr)} - \frac{1}{2}f\sqrt{(1+ff)} + \frac{1}{2}e\sqrt{(1+ee)},$$

quae ad praecedentem addita praebet

$$\text{Arc.}\,ps - 3\,\text{Arc.}\,ef = \frac{1}{2}s\sqrt{(1+ss)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{3}{2}f\sqrt{(1+ff)} + \frac{3}{2}e\sqrt{(1+ee)}.$$

Atque hoc modo si ulterius progrediamur sitque z punctum post n huiusmodi operationes inventum, erit

$$\frac{z+\sqrt{(1+zz)}}{p+\sqrt{(1+pp)}} = \left(\frac{f+\sqrt{(1+ff)}}{e+\sqrt{(1+ee)}}\right)^n,$$

unde immediate punctum z reperietur, ita ut sit

$$\text{Arc.}\,pz - n\,\text{Arc.}\,ef = \frac{1}{2}z\sqrt{(1+zz)} - \frac{1}{2}p\sqrt{(1+pp)} - \frac{n}{2}f\sqrt{(1+ff)} + \frac{n}{2}e\sqrt{(1+ee)},$$

sicque arcus pz est inventus a dato puncto p abscissus, qui arcum ef n vicibus sumtum superat quantitate geometrica.

COROLLARIUM 1

75. Quodcunque ergo multiplum arcus ef proponatur, cuius multipli exponens sit numerus n, sive is sit integer sive fractus, a dato puncto p semper abscindi poterit arcus pz, qui hoc multiplum excedat quantitate geometrice assignabili; erit enim

$$\sqrt{(1+zz)} + z = (\sqrt{(1+pp)} + p)(\sqrt{(1+ff)} + f)^n(\sqrt{(1+ee)} - e)^n$$

et

$$\sqrt{(1+zz)} - z = (\sqrt{(1+pp)} - p)(\sqrt{(1+ff)} - f)^n(\sqrt{(1+ee)} + e)^n.$$

COROLLARIUM 2

76. Quodsi ergo ad abbreviandum ponatur

$$\sqrt{(1+ee)} + e = E, \quad \sqrt{(1+ff)} + f = F, \quad \sqrt{(1+pp)} + p = P,$$

erit

$$\sqrt{(1+zz)} + z = \frac{PF^n}{E^n} \quad \text{et} \quad \sqrt{(1+zz)} - z = \frac{E^n}{PF^n},$$

unde oritur

$$\sqrt{(1+zz)} = \frac{P^2F^{2n} + E^{2n}}{2PE^nF^n} \quad \text{et} \quad z = \frac{P^2F^{2n} - E^{2n}}{2PE^nF^n}.$$

COROLLARIUM 3

77. Hinc ergo fiet

$$\frac{1}{2} z \sqrt{(1+zz)} = \frac{P^4 F^{4n} - E^{4n}}{8 P^2 E^{2n} F^{2n}}.$$

Quia tum simili modo est

$$\frac{1}{2} e \sqrt{(1+ee)} = \frac{E^4-1}{8EE}, \quad \frac{1}{2} f \sqrt{(1+ff)} = \frac{F^4-1}{8FF} \quad \text{et} \quad \frac{1}{2} p \sqrt{(1+pp)} = \frac{P^4-1}{8PP},$$

erit

$$\text{Arc.}\, pz - n\, \text{Arc.}\, ef = \frac{P^4 F^{4n} - E^{4n}}{8P^2 E^{2n} F^{2n}} - \frac{P^4-1}{8PP} - \frac{n(F^4-1)}{8FF} + \frac{n(E^4-1)}{8EE}.$$

COROLLARIUM 4

78. Si huius expressionis partes binae in unam congregentur, reperietur ista differentia geometrica

$$\text{Arc.}\, pz - n\, \text{Arc.}\, ef = \frac{(F^{2n} - E^{2n})(P^4 F^{2n} + E^{2n})}{8P^2 E^{2n} F^{2n}} - \frac{n(FF-EE)(EEFF+1)}{8EEFF}.$$

COROLLARIUM 5

79. Quemadmodum hic ex puncto dato p alterum punctum z determinavimus, ita vicissim, si punctum z pro dato accipiatur, antrorsum progrediendo simili modo punctum p ex eadem aequatione reperietur, ita ut Arc. pz superet arcum ef n vicibus sumtum quantitate geometrice assignabili.

PROBLEMA 4

80. *Dato in parabola arcu quocunque ef invenire alium arcum pz, qui se habeat ad illum in data ratione $n:1$, ita ut sit Arc. $pz = n$ Arc. ef.*

SOLUTIO

Retentis iisdem denominationibus, quibus in probl. praecedenti eiusque coroll. 2 usi sumus, quoniam fieri debet

$$\text{Arc.}\, pz - n\, \text{Arc.}\, ef = 0,$$

quantitas illa algebraica, cui haec arcuum differentia aequalis est inventa, in nihilum abire debet. Habebimus ergo ex corollario 4 hanc aequationem

$$F^{2n}P^4 + E^{2n} = \frac{nE^{2n-2}F^{2n-2}(FF-EE)(EEFF+1)}{F^{2n}-E^{2n}}P^2.$$

Ponamus brevitatis gratia $\frac{F}{E} = C$ eritque

$$C^{2n}P^4 + 1 = \frac{nC^{2n-2}(CC-1)(CCE^4+1)}{(C^{2n}-1)EE}PP,$$

unde fit

$$C^nP^2 = \frac{nC^{n-2}(CC-1)(CCE^4+1)}{2(C^{2n}-1)EE} - \sqrt{\left(\frac{nnC^{2n-4}(CC-1)^2(CCE^4+1)^2}{4(C^{2n}-1)^2E^4} - 1\right)}$$

ideoque

$$P = \sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{2(C^{2n}-1)CCEE} - \sqrt{\left(\frac{nn(CC-1)^2(CCE^4+1)^2}{4(C^{2n}-1)^2C^4E^4} - \frac{1}{C^{2n}}\right)}\right)}$$

sive

$$P = \sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE} + \frac{1}{2C^n}\right)} - \sqrt{\left(\frac{n(CC-1)(CCE^4+1)}{4(C^{2n}-1)CCEE} - \frac{1}{2C^n}\right)}.$$

Deinde si pari modo ponatur $\sqrt{(1+zz)} + z = Z$, erit $Z = C^nP$. Ex inventis autem quantitatibus P et Z ita eliciuntur ipsae p et z, ut sit

$$p = \frac{PP-1}{2P} \quad \text{et} \quad z = \frac{ZZ-1}{2Z}.$$

Restituto autem pro C valore $\frac{F}{E}$ si ponamus

$$\sqrt{\left(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})} + \frac{1}{2E^nF^n}\right)} = M,$$

$$\sqrt{\left(\frac{n(FF-EE)(EEFF+1)}{4EEFF(F^{2n}-E^{2n})} - \frac{1}{2E^nF^n}\right)} = N,$$

reperietur

$$P = E^n(M-N) \quad \text{et} \quad \frac{1}{P} = F^n(M+N),$$

$$Z = F^n(M-N) \quad \text{et} \quad \frac{1}{Z} = E^n(M+N),$$

unde concluduntur ipsae abscissae

$$p = -\tfrac{1}{2}M(F^n - E^n) - \tfrac{1}{2}N(F^n + E^n),$$

$$z = +\tfrac{1}{2}M(F^n - E^n) - \tfrac{1}{2}N(F^n + E^n).$$

Cum autem M et N tam affirmative quam negative accipere liceat, capiatur N negativum, ut punctum z in istum parabolae ramum incidat, in quo est arcus ef, eritque

$$p = \frac{1}{2}N(F^n + E^n) - \frac{1}{2}M(F^n - E^n),$$

$$z = \frac{1}{2}N(F^n + E^n) + \frac{1}{2}M(F^n - E^n).$$

Ex quibus formulis si definiantur puncta p et z, erit

$$\text{Arc. } pz = n \text{ Arc. } ef.$$

COROLLARIUM 1

81. Ambae ergo abscissae $AP = p$ et $AZ = z$ ita sunt comparatae, ut sit

$$z + p = N(F^n + E^n) \quad \text{et} \quad z - p = M(F^n - E^n).$$

Hinc erit valoribus pro M et N restituendis

$$pz = \frac{nE^nF^n(F^2 - E^2)(EEFF + 1)}{4EEFF(F^{2n} - E^{2n})} - \frac{F^{2n} + E^{2n}}{4E^nF^n}$$

et

$$pp + zz = \frac{n(F^2 - E^2)(EEFF + 1)(F^{2n} + E^{2n})}{4EEFF(F^{2n} - E^{2n})} - 1.$$

COROLLARIUM 2

82. Si sit $n = 1$, erit

$$M = \sqrt{\left(\frac{EEFF + 1}{4EEFF} + \frac{1}{2EF}\right)} = \frac{EF + 1}{2EF} \quad \text{et} \quad N = \frac{EF - 1}{2EF},$$

unde fit

$$z + p = \frac{1}{2}F + \frac{1}{2}E - \frac{1}{2E} - \frac{1}{2F} \quad \text{et} \quad z - p = \frac{1}{2}F - \frac{1}{2}E + \frac{1}{2E} - \frac{1}{2F}$$

ideoque

$$2p = E - \frac{1}{E} \quad \text{seu} \quad p = e \quad \text{et} \quad 2z = F - \frac{1}{F} \quad \text{seu} \quad z = f,$$

puncta scilicet p et z in puncta e et f incidunt.

COROLLARIUM 3

83. Si arcus pz debeat esse duplus arcus dati ef seu $n=2$, erit

$$M=\sqrt{\left(\frac{EEFF+1}{2EEFF(FF+EE)}+\frac{1}{2EEFF}\right)}=\sqrt{\frac{(EE+1)(FF+1)}{2EEFF(EE+FF)}}$$

et

$$N=\sqrt{\left(\frac{EEFF+1}{2EEFF(FF+EE)}-\frac{1}{2EEFF}\right)}=\sqrt{\frac{(EE-1)(FF-1)}{2EEFF(EE+FF)}};$$

unde, si arcus ef in vertice A terminetur, ut sit $e=0$ et $E=1$, erit $M=\frac{1}{F}$ et $N=0$; sicque prodit $z+p=0$ et $z-p=\frac{FF-1}{F}=2f$; ideoque $p=-f$ et $z=+f$. Hoc ergo casu arcus pz medium in verticem A incidit et utrinque arcum ipsi ef seu Af aequalem complectitur.

COROLLARIUM 4

84. Si arcus pz debeat esse triplus arcus ef seu $n=3$, erit

$$M=\sqrt{\left(\frac{3(EEFF+1)}{4EEFF(F^4+E^2F^2+E^4)}+\frac{1}{2E^3F^3}\right)}$$

sive

$$M=\sqrt{\frac{3E^3F^3+3EF+2F^4+2EEFF+2E^4}{4E^3F^3(F^4+EEFF+E^4)}}$$

et

$$N=\sqrt{\frac{3E^3F^3+3EF-2F^4-2EEFF-2E^4}{4E^3F^3(F^4+EEFF+E^4)}}.$$

COROLLARIUM 5

85. Si hoc casu, quo $n=3$, arcus ef in vertice A incipiat, erit $e=0$ et $E=1$, unde

$$M=\sqrt{\frac{2F^4+3F^3+2FF+3F+2}{4F^3(F^4+F^2+1)}}$$

sive

$$M=(F+1)\sqrt{\frac{2FF-F+2}{4F^3(F^4+F^2+1)}}$$

et

$$N=(F-1)\sqrt{\frac{-2FF-F-2}{4F^3(F^4+F^2+1)}},$$

qui ergo valor est imaginarius.

COROLLARIUM 6

86. Ut ergo arcus ef triplum exhiberi possit, is non in vertice A terminari potest seu E debet esse maius quam 1 atque adeo limes dabitur, infra quem accipi nequeat. Ad quem limitem inveniendum resolvi oportet hanc aequationem

$$3E^3F^3 + 3EF = 2F^4 + 2EEFF + 2E^4.$$

In hunc finem ponatur $EF = S$ et $EE + FF = R$; erit

$$3S^3 + 3S = 2RR - 2SS \quad \text{ideoque} \quad R = \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)},$$

unde fit

$$F + E = \sqrt{\left(2S + \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)},$$

$$F - E = \sqrt{\left(-2S + \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)}.$$

Et cum sit $E > 1$ et $F > 1$, debet esse $R > 2$ et $3S^3 + 2SS + 3S > 8$ ideoque $S > 1$.

COROLLARIUM 7

87. Generatim ergo pro casu $n = 3$ oportet sit

$$3S^3 + 3S > 2RR - 2SS \quad \text{ideoque} \quad R < \sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)};$$

quare si α sit numerus unitate minor, reperitur

$$F + E = \sqrt{\left(2S + \alpha\sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)},$$

$$F - E = \sqrt{\left(-2S + \alpha\sqrt{\left(\tfrac{3}{2}S^3 + SS + \tfrac{3}{2}S\right)}\right)}.$$

Debet ergo esse $\alpha\alpha > \frac{8S}{3SS + 2S + 3}$ et $S > 1$.

COROLLARIUM 8

88. Ponamus $S = 2$; erit $\alpha\alpha > \frac{16}{19}$. Capiatur $\alpha = 1$, ut sit $EF = 2$ et $EE + FF = \sqrt{19}$; erit

$$F + E = \sqrt{(\sqrt{19} + 4)}, \quad E = \tfrac{1}{2}\sqrt{(\sqrt{19} + 4)} - \tfrac{1}{2}\sqrt{(\sqrt{19} - 4)},$$

$$F - E = \sqrt{(\sqrt{19} - 4)}, \quad F = \tfrac{1}{2}\sqrt{(\sqrt{19} + 4)} + \tfrac{1}{2}\sqrt{(\sqrt{19} - 4)};$$

ergo

$$e = \tfrac{1}{8}\sqrt{(\sqrt{19}+4)} - \tfrac{3}{8}\sqrt{(\sqrt{19}-4)}$$

et

$$f = \tfrac{1}{8}\sqrt{(\sqrt{19}+4)} + \tfrac{3}{8}\sqrt{(\sqrt{19}-4)}.$$

Porro reperitur

$$M = \frac{1}{2\sqrt{2}} \quad \text{et} \quad N = 0;$$

unde

$$z = -p = \frac{1}{4\sqrt{2}}\left(2+\sqrt{19}\right)\sqrt{(\sqrt{19}-4)};$$

hic ergo arcus triplus utrinque circa verticem aequaliter extenditur.

III. DE COMPARATIONE SUPERFICIERUM SPHAEROIDIS ELLIPTICI COMPRESSI ET CONOIDIS HYPERBOLICI

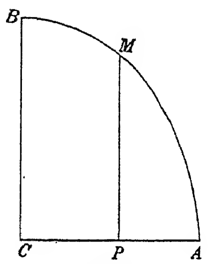

Fig. 4.

89. Sit igitur primum propositum sphaeroides ellipticum genitum rotatione ellipsis BMA (Fig. 4) circa axem minorem AC. Ponatur semiaxis minor $CA = a$ et semiaxis maior $CB = a\sqrt{m}$ existente m numero unitate maiori. Sumta iam in axe minore a centro C abscissa $CP = x$ erit applicata $PM = \sqrt{m(aa-xx)}$, unde elementum ellipticum $= dx\sqrt{\frac{aa+(m-1)xx}{aa-xx}}$.

90. Posita nunc ratione diametri ad peripheriam $= 1:\pi$ erit portio superficiei sphaeroidicae a revolutione arcus BM genita, seu quae respondet abscissae $CP = x$, aequalis huic integrali $2\pi\int dx\sqrt{m(aa+(m-1)xx)}$. Indicetur hoc integrale, quod tanquam functio abscissae x spectetur, hoc modo

$$\int dx\sqrt{m(aa+(m-1)xx)} = \Pi . x.$$

91. Portio ergo superficiei sphaeroidicae ellipticae abscissae $CP = x$ respondens erit $= 2\pi \cdot \Pi . x$, ubi functio $\Pi . x$, uti perspicuum est, a logarithmis seu rectificatione parabolae pendet, eritque $\Pi . x = 0$, si $x = 0$; sin autem ponatur $x = a$, tum $2\pi \cdot \Pi . a$ exhibebit semissem totius superficiei sphaeroidis.

Sit porro conoides hyperbolicum genitum revolutione hyperbolae *am* p. 145) circa suum axem *cap*, cuius centrum sit in *c*. Ponatur eius

semiaxis transversus $ca = c$, semiaxis autem coniugatus $= c\sqrt{n}$. Sumta ergo in axe a centro c abscissa quacunque $cp = y$, quae quidem sit $> c$, erit applicata $pm = \sqrt{n(yy - cc)}$ et elementum hyperbolicum $= dy\sqrt{\frac{(n+1)yy - cc}{yy - cc}}$.

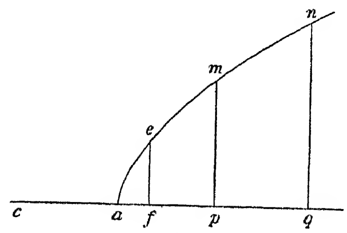

Fig. 5.

93. Hinc erit portio superficiei conoidis istius hyperbolici ex arcu am genita seu abscissae $cp = y$ respondens $= 2\pi\int dy\sqrt{n((n+1)yy - cc)}$. Quod integrale cum spectari possit tanquam functio ipsius y, ita indicetur

$$\int dy\sqrt{n((n+1)yy - cc)} = \Theta . y$$

fitque $\Theta . y = 0$, si capiatur $y = c$. Erit ergo superficies conoidis hyperbolici abscissae $cp = y$ respondens $= 2\pi \cdot \Theta . y$.

94. Comparentur hae binae formulae cum illis, quae supra § 38 sunt expositae, et cum sit

$$\Pi . x = \int \frac{dx(aa + (m-1)xx)\sqrt{m}}{\sqrt{(aa + (m-1)xx)}},$$

erit

$$A = aa, \quad C = m - 1,$$

$$\mathfrak{A}\sqrt{(m-1)} = aa\sqrt{m} \quad \text{et} \quad \frac{m-1}{aa}\mathfrak{B}\sqrt{(m-1)} = (m-1)\sqrt{m};$$

unde fit

$$\mathfrak{A} = \frac{aa\sqrt{m}}{\sqrt{(m-1)}} \quad \text{et} \quad \mathfrak{B} = \frac{aa\sqrt{m}}{\sqrt{(m-1)}}.$$

95. Deinde pro hyperbola cum sit

$$\Theta . y = \int \frac{dy(-cc + (n+1)yy)\sqrt{n}}{\sqrt{(-cc + (n+1)yy)}},$$

fiat $E = -cc$ et $F = n + 1$ eritque ob $\mathfrak{C} = 0$

$$-\int \frac{dy\left(\mathfrak{A} + \frac{F}{E}\mathfrak{B}yy\right)\sqrt{F}}{\sqrt{(E + Fyy)}} = \frac{aa\sqrt{m(n+1)}}{\sqrt{(m-1)}}\int \frac{dy\left(-1 + \frac{(n+1)yy}{cc}\right)}{\sqrt{(-cc + (n+1)yy)}},$$

ergo

$$-\int \frac{dy\left(\mathfrak{A}+\frac{F}{E}\mathfrak{B}yy\right)\sqrt{F}}{\sqrt{(E+Fyy)}} = \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\,\Theta.y.$$

96. His ergo substitutionibus factis habebimus hanc aequationem

$$\Pi.x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\,\Theta.y=\text{Const.}+\frac{(n+1)\sqrt{m(m-1)}}{cc}\,kxy,$$

cui satisfacit haec relatio inter x et y

$$\frac{aay}{m-1}=k\sqrt{\left(\frac{aa}{m-1}+xx\right)}-x\sqrt{\left(kk-\frac{aacc}{(m-1)(n+1)}\right)}$$

seu

$$\frac{ccx}{n+1}=k\sqrt{\left(-\frac{cc}{n+1}+yy\right)}+y\sqrt{\left(kk-\frac{aacc}{(m-1)(n+1)}\right)},$$

ubi $\sqrt{\left(kk-\frac{aacc}{(m-1)(n+1)}\right)}$ negative accipi conveniet.

97. Vel ponatur $k=\frac{ae}{\sqrt{(m-1)}}$, et si fuerit

$$y=\frac{e}{a}\sqrt{(aa+(m-1)xx)}+\frac{x\sqrt{(m-1)}}{a\sqrt{(n+1)}}\sqrt{((n+1)ee-cc)}$$

seu

$$x=\frac{ae\sqrt{(n+1)}}{cc\sqrt{(m-1)}}\sqrt{((n+1)yy-cc)}-\frac{ay\sqrt{(n+1)}}{cc\sqrt{(m-1)}}\sqrt{((n+1)ee-cc)},$$

erit

$$\Pi.x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\,\Theta.y=\text{Const.}+\frac{(n+1)ae\sqrt{m}}{cc}xy.$$

98. Ad constantem autem definiendam ponatur $x=0$, ut sit $\Pi.x=0$, eritque $y=e$, unde prodit

$$\text{Const.}=\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}\,\Theta.e;$$

sicque habebitur

$$\Pi.x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta.y-\Theta.e)=\frac{(n+1)ae\sqrt{m}}{cc}xy.$$

At si in hyperbola capiatur abscissa $cf=e$, erit superficies conoidis ex arcu em nata $=2\pi\cdot(\Theta.y-\Theta.e)$.

99. Quoniam igitur y per x determinatur, erit quoque

$$\sqrt{((n+1)yy-cc)} = \frac{e}{a}x\sqrt{(m-1)(n+1)} + \frac{1}{a}\sqrt{(aa+(m-1)xx)((n+1)ee-cc)},$$

unde fit

$$y + \delta\sqrt{((n+1)yy-cc)} = \left(\frac{e}{a} + \frac{\delta}{a}\sqrt{((n+1)ee-cc)}\right)\sqrt{(aa+(m-1)xx)}$$
$$+ x\left(\frac{\delta e}{a}\sqrt{(m-1)(n+1)} + \frac{\sqrt{(m-1)}}{a\sqrt{(n+1)}}\sqrt{((n+1)ee-cc)}\right);$$

sit

$$1 : \delta\sqrt{(m-1)(n+1)} = \delta : \frac{\sqrt{(m-1)}}{\sqrt{(n+1)}};$$

erit

$$\delta = \frac{1}{\sqrt{(n+1)}}$$

hincque obtinetur

$$\sqrt{((n+1)yy-cc)} + y\sqrt{(n+1)}$$
$$= \left(\frac{e\sqrt{(n+1)}}{a} + \frac{1}{a}\sqrt{((n+1)ee-cc)}\right)\left(\sqrt{(aa+(m-1)xx)} + x\sqrt{(m-1)}\right).$$

100. Datis ergo abscissis $CP = x$ et $cf = e$ abscissa $cp = y$ ita definiri debet, ut sit

$$\frac{\sqrt{((n+1)yy-cc)} + y\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)} + e\sqrt{(n+1)}} = \sqrt{\left(1 + \frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}.$$

Deinde autem est

$$exy = \frac{ay\sqrt{((n+1)yy-cc)}}{2\sqrt{(m-1)(n+1)}} - \frac{ae\sqrt{((n+1)ee-cc)}}{2\sqrt{(m-1)(n+1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{2a(n+1)}.$$

PROBLEMA HUGENIANUM

101. *Dato sphaeroide elliptico lato ABC invenire conoides hyperbolicum apm, ita ut circulus describi possit geometrice, cuius area aequalis sit futura utrique superficiei sphaeroidicae et conoidicae iunctim sumtae.*[1]

1) Vide notam 1 p. 111. A. K.

SOLUTIO PRIMA

Manentibus pro utroque corpore denominationibus modo expositis statuatur

$$\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}=1 \quad \text{seu} \quad cc=\frac{aa\sqrt{m(n+1)}}{\sqrt{n(m-1)}},$$

unde semiaxis transversus hyperbolae c determinatur numero n seu eius specie arbitrio nostro relicta, eritque stabilita superiori relatione inter x et y

$$\Pi.x+(\Theta.y-\Theta.e)=\frac{(n+1)ae\sqrt{m}}{cc}xy=\frac{exy\sqrt{n(m-1)(n+1)}}{a}.$$

102. Cum nunc sit superficies sphaeroidis ex arcu BM nata seu Sup. $BM=2\pi\cdot\Pi.x$ et superficies conoidis ex arcu em nata seu Sup. $em=2\pi(\Theta.y-\Theta.e)$, erit

$$\text{Sup.}\,BM+\text{Sup.}\,em=\frac{2\pi exy\sqrt{n(m-1)(n+1)}}{a}.$$

Unde si hae duae superficies iunctim sumtae aequentur circulo, cuius radius $=r$, ob eius aream $=\pi rr$ erit

$$rr=\frac{2\,exy\sqrt{n(m-1)(n+1)}}{a}.$$

103. Hic iam continetur solutio problematis sensu multo latiori accepti. Casu enim HUGENIANO, quo integrum sphaeroides assumitur seu, quod eodem redit, eius semissis, erit $x=a$; tum vero punctum e in vertice a capi oportet, unde fit $e=c$. Erit ergo hoc casu

$$y=c\sqrt{m}+\frac{c\sqrt{n(m-1)}}{\sqrt{(n+1)}}=cp$$

fietque

$$\text{Sup.}\,BA+\text{Sup.}\,am=2\pi(n+1)aa\cdot\frac{m+\sqrt{mn(n-1)}}{\sqrt{(n+1)}}.$$

104. Radio ergo circuli utrique superficiei simul aequalis posito $=r$ erit

$$rr=2aa(m(n+1)+\sqrt{mn(m-1)(n+1)})$$

sive

$$r=a\sqrt{2\left(\sqrt{m(n+1)}+\sqrt{n(m-1)}\right)\sqrt{m(n+1)}}.$$

Atque erit

$$cp = y = \frac{c}{\sqrt{(n+1)}}(\sqrt{m(n+1)} + \sqrt{n(m-1)});$$

tum vero accipi debet

$$c = a\sqrt[4]{\frac{m(n+1)}{n(m-1)}}.$$

Quae est solutio simplicissima Problematis HUGENIANI.

SOLUTIO SECUNDA

105. Cum relatio inter x et y sit ita comparata, ut sit

$$\frac{\sqrt{((n+1)yy-cc)}+y\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)}+e\sqrt{(n+1)}} = \sqrt{\left(1+\frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}$$

sitque

$$\Pi.x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta.y - \Theta.e)$$

$$= \frac{\sqrt{m(n+1)}}{2cc}\left(\frac{aay\sqrt{((n+1)yy-cc)}}{\sqrt{(m-1)}} - \frac{aae\sqrt{((n+1)ee-cc)}}{\sqrt{(m-1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right),$$

capiatur in conoide nova abscissa $cq = z$ et pro e iam sumatur y, ut sit

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{\sqrt{((n+1)yy-cc)}+y\sqrt{(n+1)}} = \sqrt{\left(1+\frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)};$$

erit pariter

$$\Pi.x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta.z - \Theta.y)$$

$$= \frac{\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}} - \frac{aay\sqrt{((n+1)yy-cc)}}{\sqrt{(m-1)}} + \frac{ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right).$$

106. Addantur hae formulae invicem atque y prorsus eliminabitur; fiet enim

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)}+e\sqrt{(n+1)}} = \left(\sqrt{\left(1+\frac{(m-1)xx}{aa}\right)} + \frac{x}{a}\sqrt{(m-1)}\right)^2$$

eritque

$$2\Pi.x + \frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta.z - \Theta.e)$$

$$= \frac{\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}} - \frac{aae\sqrt{((n+1)ee-cc)}}{\sqrt{(m-1)}} + \frac{2ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right).$$

107. Statuatur iam

$$\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}=2 \quad \text{seu} \quad cc=\frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}};$$

erit per $\frac{2\pi}{2}$ multiplicando

$$\text{Sup. } BM+\text{Sup. } en$$
$$=\frac{\pi\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}}-\frac{aae\sqrt{((n+1)ee-cc)}}{\sqrt{(m-1)}}+\frac{2ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right),$$

unde facile radius circuli aequalis definitur.

108. Sit nunc pro casu HUGENIANO $x=a$ et $e=c$; erit

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{c\,(\sqrt{n}+\sqrt{(n+1)})}=\left(\sqrt{m}+\sqrt{(m-1)}\right)^2.$$

Hincque invento z existenteque

$$cc=\frac{aa\sqrt{m(n+1)}}{2\sqrt{n(m-1)}}$$

erit

$$\text{Sup. } BA+\text{Sup. } an=\frac{\pi\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}}-\frac{aacc\sqrt{n}}{\sqrt{(m-1)}}+\frac{2aacc\sqrt{m}}{\sqrt{(n+1)}}\right).$$

SOLUTIO GENERALIS

109. Si hac ratione continuo ulterius progrediamur, ut supra pro parabola est factum, reperietur, si abscissa $cq=z$ existente $cf=e$ ita capiatur, ut sit

$$\frac{\sqrt{((n+1)zz-cc)}+z\sqrt{(n+1)}}{\sqrt{((n+1)ee-cc)}+e\sqrt{(n+1)}}=\left(\sqrt{\left(1+\frac{(m-1)xx}{aa}\right)}+\frac{x}{a}\sqrt{(m-1)}\right)^{\mu},$$

fore

$$\mu\Pi.\,x+\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}}(\Theta.\,z-\Theta.\,e)$$
$$=\frac{\sqrt{m(n+1)}}{2cc}\left(\frac{aaz\sqrt{((n+1)zz-cc)}}{\sqrt{(m-1)}}-\frac{aae\sqrt{((n+1)ee-cc)}}{\sqrt{(m-1)}}+\frac{\mu ccx\sqrt{(aa+(m-1)xx)}}{\sqrt{(n+1)}}\right)$$
$$=\frac{\mu}{2\pi}\text{ Sup. } BM+\frac{aa\sqrt{m(n+1)}}{2\pi cc\sqrt{n(m-1)}}\text{ Sup. } en.$$

110. Pro casu ergo HUGENII posito $x = a$ et $e = c$ fiat $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = \mu$ et capiatur abscissa $cq = z$, ita ut sit

$$\frac{\sqrt{((n+1)zz - cc)} + z\sqrt{(n+1)}}{c(\sqrt{n} + \sqrt{(n+1)})} = \left(\sqrt{m} + \sqrt{(m-1)}\right)^{\mu},$$

eritque

$$\text{Sup. } BA + \text{Sup. } an = \frac{\pi\sqrt{m(n+1)}}{\mu cc}\left(\frac{aaz\sqrt{((n+1)zz - cc)}}{\sqrt{(m-1)}} - \frac{aacc\sqrt{n}}{\sqrt{(m-1)}} + \frac{\mu aacc\sqrt{m}}{\sqrt{(n+1)}}\right)$$

sive

$$\text{Sup. } BA + \text{Sup. } an = \pi\left(z\sqrt{n((n+1)zz - cc)} - ncc + \frac{\mu cc\sqrt{mn(m-1)}}{\sqrt{(n+1)}}\right)$$
$$= \pi\left(z\sqrt{n((n+1)zz - cc)} - ncc + maa\right).$$

111. Quaecunque ergo fuerit hyperbola, ex qua conoides nascitur, dummodo sit $\frac{aa\sqrt{m(n+1)}}{cc\sqrt{n(m-1)}} = \mu$ numerus rationalis, ab eo semper portio an abscindi poterit, cuius superficies ad superficiem sphaeroidis BMA addita per circulum exhiberi potest, cuius radius r geometrice est assignabilis; erit enim

$$r = \sqrt{\left(maa - ncc + z\sqrt{n((n+1)zz - cc)}\right)}.$$

112. Quo autem facilius pateat, quomodo abscissa $cq = z$ reperiri debeat, cum sit

$$\sqrt{\left(\frac{(n+1)zz}{cc} - 1\right)} + \frac{z}{c}\sqrt{(n+1)} = \left(\sqrt{(n+1)} + \sqrt{n}\right)\left(\sqrt{m} + \sqrt{(m-1)}\right)^{\mu},$$

erit

$$\frac{z}{c}\sqrt{(n+1)} - \sqrt{\left(\frac{(n+1)zz}{cc} - 1\right)} = \left(\sqrt{(n+1)} - \sqrt{n}\right)\left(\sqrt{m} - \sqrt{(m-1)}\right)^{\mu};$$

hinc facile tam z quam $\sqrt{((n+1)zz - cc)}$ colligentur.

113. Hinc autem porro concluditur fore

$$z\sqrt{n((n+1)zz - cc)} = \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}\left(\sqrt{(n+1)} + \sqrt{n}\right)^2\left(\sqrt{m} + \sqrt{(m-1)}\right)^{2\mu}$$
$$- \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}\left(\sqrt{(n+1)} - \sqrt{n}\right)^2\left(\sqrt{m} - \sqrt{(m-1)}\right)^{2\mu}.$$

At si ponatur brevitatis gratia

$$\sqrt{m} + \sqrt{(m-1)} = M \quad \text{et} \quad \sqrt{n} + \sqrt{(n+1)} = N,$$

erit

$$z = \frac{c}{2\sqrt{(n+1)}}(M^{\mu}N + M^{-\mu}N^{-1})$$

et

$$r = \sqrt{\left(maa + \frac{cc\sqrt{n}}{4\sqrt{(n+1)}}(M^{\mu} - M^{-\mu})(M^{\mu}N^{2} + M^{-\mu}N^{-2})\right)}$$

sicque problema non difficulter construitur, dummodo exponens μ fuerit rationalis.

114. Haec igitur exempla sufficiant usum novae methodi, quam adumbravi, ostendisse; etsi enim haec eadem exempla methodo consueta iam sint soluta, tamen non solum ad calculos admodum intricatos deveniri solet, sed etiam integratione, qua formulae differentiales vel ad quadraturam circuli vel ad logarithmos reducantur, absolute est opus. Huius igitur novae methodi insigne commodum in hoc consistit, quod eius beneficio eadem problemata tam sine laborioso calculo quam sine ulla integratione resolvi queant; quam ob causam inde merito multo maiora ac sublimiora expectare licet, quae vim omnium consuetarum methodorum penitus superent.

SPECIMEN ALTERUM METHODI NOVAE QUANTITATES TRANSCENDENTES INTER SE COMPARANDI DE COMPARATIONE ARCUUM ELLIPSIS

Commentatio 261 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p. 3—48
Summarium (Commentationum 261 et 263) ibidem p. 5—8[1])

1. Primum huius methodi specimen, quod nuper[2]) exhibui, in comparatione arcuum circuli et parabolae conicae versabatur; quae comparatio etsi in se spectata non est nova, cum methodis vulgaribus iam pridem sit expedita, tamen inde exordiendum est visum, quo novae huius methodi, quam adumbravi, vis melius perspiciatur; quod non solum ad easdem veritates, quae methodis consuetis erui solent, perducat, sed etiam viam longe faciliorem et expeditiorem eadem praestandi patefaciat. Methodus enim consueta integrationes satis taediosas requirit atque ita est comparata, ut, nisi arcus istarum curvarum, qui inter se sunt comparandi, ad quadraturas cognitas circuli ac hyperbolae revocari potuissent, nullo modo in subsidium vocari potuisset.

2. Quantum ergo haec nova methodus praestare valeat, uberius ex comparatione arcuum ellipsis et hyperbolae perspicietur; quarum curvarum rectificatio cum nullo modo neque ad circuli quadraturam neque ad logarithmos reduci queat, methodis consuetis nullus amplius locus relinquitur neque per eas modus patet diversos istarum curvarum arcus inter se conferendi. Quare

1) Vide p. 108. A. K.

2) L. EULERI Commentatio 263 (indicis ENESTROEMIANI); vide p. 108. A. K.

cum ostendero novae huius methodi beneficio comparationem arcuum elliptìcorum et hyperbolicorum pari cum successu institui posse atque arcuum parabolicorum, quoniam methodi vulgares ad id plane sunt ineptae, eo magis summus usus novae methodi inde elucebit.

3. Inveni autem huius methodi ope arcus tam ellipticos quam hyperbolicos pari modo inter se comparari posse atque arcus parabolicos neque id impedimento esse, quod harum curvarum rectificatio vires Analyseos penitus transgredi videatur. Quin etiam haec comparatio sub iisdem conditionibus atque in parabola institui potest, ita ut proposito sive in ellipsi sive in hyperbola arcu quocunque ab alio quovis eiusdem curvae puncto arcus abscindi possit, qui ab illo differat quantitate geometrice assignabili. Simili autem modo a puncto quovis arcus exhiberi poterit, qui ab arcu proposito vel bis vel ter vel toties sumto, quoties lubuerit, quantitate geometrica discrepet.

4. Porro autem effici potest, ut haec differentia plane in nihilum abeat arcusque inventus ipsi arcui proposito eiusve multiplo adeo fiat aequalis, perinde atque in parabola id fieri posse notum est. Similiter quidem usu venit, ut bini arcus aequales exhiberi nequeant, qui non simul inter se sint similes; verum hoc multo magis notatu erit dignum, quod tam in ellipsi quam hyperbola proposito arcu quocunque semper alius arcus assignari queat, qui illius duplo vel triplo vel multiplo cuicunque sit aequalis.

5. Quemadmodum igitur ratione comparationis diversorum arcuum ellipsis et hyperbola indolem parabolae sequuntur, ita curva lemniscata ipsi circulo similis deprehenditur. In ea enim curva aeque ac in circulo si propositus fuerit arcus quicunque, a puncto quovis dato arcum abscindere licet, qui proposito vel fuerit aequalis vel duplo maior vel triplo vel toties, quoties lubuerit. In hac namque curva perinde atque in circulo eiusmodi arcus non dantur, quorum differentia geometrice possit assignari.

6. Quae autem hic sum allaturus, multo latius patent quam ad curvas commemoratas, ellipsin, hyperbolam et lemniscatam, quippe quae tantum casus quasi simplicissimos constituunt formularum, quas haec methodus suppeditat. His enim formulis evolutis similem comparationem in infinitis aliis curvarum generibus instituere licebit. Quemadmodum autem primum specimen

evolutione huius aequationis innitebatur

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy,$$

ita hic aequationem latius patentem fundamenti loco assumi oportet, ex qua tamen utraque variabilis ope extractionis radicis quadratae definiri queat. Sit igitur proposita haec

AEQUATIO CANONICA

$$0 = \alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy$$

7. Quodsi ex hac aequatione tam valorem ipsius x quam ipsius y seorsim extrahamus, obtinebimus

$$y = \frac{-\delta x + \sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))}}{\gamma + \zeta xx},$$

$$x = \frac{-\delta y - \sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))}}{\gamma + \zeta yy},$$

ubi signis radicalibus diversa tribuimus signa, quoniam ab arbitrio nostro pendent, dummodo eorum in sequentibus debita ratio teneatur.

8. Ponamus, ut brevitati consulamus, has formulas surdas

$$\sqrt{(\delta\delta xx - (\alpha + \gamma xx)(\gamma + \zeta xx))} = X$$

et

$$\sqrt{(\delta\delta yy - (\alpha + \gamma yy)(\gamma + \zeta yy))} = Y,$$

ut habeamus

$$y = \frac{-\delta x + X}{\gamma + \zeta xx} \quad \text{seu} \quad X = \gamma y + \delta x + \zeta xxy,$$

$$x = \frac{-\delta y - Y}{\gamma + \zeta yy} \quad \text{seu} \quad -Y = \gamma x + \delta y + \zeta xyy.$$

9. Nunc aequatio canonica etiam differentietur eritque

$$0 = dx(\gamma x + \delta y + \zeta xyy) + dy(\gamma y + \delta x + \zeta xxy),$$

unde colligimus fore

$$0 = -Ydx + Xdy \quad \text{sive} \quad \frac{dy}{Y} - \frac{dx}{X} = 0.$$

Cum igitur X sit functio ipsius x et Y ipsius y, erit integrando

$$\int\frac{dy}{Y}-\int\frac{dx}{X}=\text{Const.}$$

10. Vicissim ergo novimus, si huiusmodi aequatio integralis fuerit proposita

$$\int\frac{dy}{Y}-\int\frac{dx}{X}=\text{Const.},$$

in qua X et Y eiusmodi functiones irrationales ipsarum x et y designent, ut sit

$$X=\sqrt{(\delta\delta xx-(\alpha+\gamma xx)(\gamma+\zeta xx))}$$

et

$$Y=\sqrt{(\delta\delta yy-(\alpha+\gamma yy)(\gamma+\zeta yy))},$$

tum huic aequationi satisfacere relationem inter x et y per aequationem canonicam definitam.

11. Quemadmodum autem invenimus aequationem $\frac{dy}{Y}-\frac{dx}{X}=0$, ita consideremus nunc aequationem latius patentem

$$\frac{Qdy}{Y}-\frac{Pdx}{X}=dV$$

et investigemus, cuiusmodi functiones P et Q esse queant ipsarum x et y, ut dV integrationem admittat ideoque differentia formularum integralium

$$\int\frac{Qdy}{Y}-\int\frac{Pdx}{X}=\text{Const.}+V$$

algebraice exhiberi queat.

13.[1]) Quo haec investigatio facilius institui queat, ponamus $xy=u$ et ob $xdy+ydx=du$ habebimus $dy=\frac{du}{x}-\frac{ydx}{x}$, qui valor loco dy in aequatione differentiali substitutus dabit

$$0=dx(\gamma x+\delta y+\zeta xyy)+\frac{du}{x}(\gamma y+\delta x+\zeta xxy)-dx\left(\frac{\gamma yy}{x}+\delta y+\zeta xyy\right)$$

seu per x multiplicando

$$0=dx(\gamma xx-\gamma yy)+du(\gamma y+\delta x+\zeta xxy)$$

seu

$$0=\gamma dx(xx-yy)+Xdu.$$

1) In editione principe loco numerum 12 et qui sequuntur falso numeri 13 et qui sequuntur scripti sunt. Falsos paragraphorum numeros retinendos esse putavimus. A. K.

14. Erit ergo $\frac{dx}{X} = \frac{du}{\gamma(yy-xx)}$, et cum sit $\frac{dy}{Y} = \frac{dx}{X}$, erit quoque $\frac{dy}{Y} = \frac{du}{\gamma(yy-xx)}$, unde habebimus

$$dV = \frac{(Q-P)du}{\gamma(yy-xx)}.$$

Primo ergo patet, si sit $Q = yy$ et $P = xx$, fore

$$dV = \frac{du}{\gamma} \quad \text{et} \quad V = \frac{u}{\gamma} = \frac{xy}{\gamma}.$$

Hinc assumta aequatione canonica erit

$$\int \frac{yy\,dy}{Y} - \int \frac{xx\,dx}{X} = \text{Const.} + \frac{xy}{\gamma}.$$

15. Similis autem integratio quantitatis V quoque succedit, si pro P et Q accipiantur potestates quaevis parium dimensionum ipsarum x et y. Quod ut appareat, ponamus $xx + yy = t$ et ob $xy = u$ aequatio canonica abit in hanc formam

$$0 = \alpha + \gamma t + 2\delta u + \zeta uu,$$

unde fit $t = \frac{-\alpha - 2\delta u - \zeta uu}{\gamma}$.

16. Ponamus iam $P = x^4$ et $Q = y^4$; erit

$$dV = \frac{du}{\gamma}(xx+yy) = \frac{t\,du}{\gamma} \quad \text{ideoque} \quad dV = \frac{-du}{\gamma\gamma}(\alpha + 2\delta u + \zeta uu);$$

unde integrando fit

$$V = \frac{-\alpha u}{\gamma\gamma} - \frac{\delta uu}{\gamma\gamma} - \frac{\zeta u^3}{3\gamma\gamma} \quad \text{sive} \quad V = \frac{-xy}{3\gamma\gamma}(3\alpha + 3\delta xy + \zeta xxyy).$$

Vel ob $\zeta xxyy = -\alpha - \gamma(xx+yy) - 2\delta xy$ habebitur

$$V = \frac{-xy}{3\gamma\gamma}(2\alpha - \gamma(xx+yy) + \delta xy).$$

17. Quare nostra aequatio canonica etiam satisfaciet huic aequationi integrali

$$\int \frac{y^4\,dy}{Y} - \int \frac{x^4\,dx}{X} = \text{Const.} - \frac{xy}{3\gamma\gamma}(3\alpha + 3\delta xy + \zeta xxyy).$$

Atque his tribus casibus colligendis aequatio canonica satisfaciet huic aequationi differentiali latius patenti

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4)}{\sqrt{(\delta\delta yy-(\alpha+\gamma yy)(\gamma+\zeta yy))}} - \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{(\delta\delta xx-(\alpha+\gamma xx)(\gamma+\zeta xx))}}$$

$$= \text{Const.} + \frac{\mathfrak{B}xy}{\gamma} - \frac{\mathfrak{C}xy}{3\gamma\gamma}(3\alpha+3\delta xy+\zeta xxyy).$$

18. Si ulterius progredi velimus, ponamus $P=x^6$ et $Q=y^6$ fietque

$$dV = \frac{du}{\gamma}(y^4+xxyy+x^4) = \frac{du}{\gamma}(tt-uu);$$

substituto ergo pro t valore invento erit

$$dV = \frac{du}{\gamma^3}\left(\alpha\alpha+4\alpha\delta u+(4\delta\delta+2\alpha\zeta-\gamma\gamma)uu+4\delta\zeta u^3+\zeta\zeta u^4\right)$$

ideoque integrando

$$V = \frac{u}{\gamma^3}\left(\alpha\alpha+2\alpha\delta u+\frac{1}{3}(4\delta\delta+2\alpha\zeta-\gamma\gamma)uu+\delta\zeta u^3+\frac{1}{5}\zeta\zeta u^4\right).$$

Unde erit per aequationem canonicam

$$\int\frac{y^6dy}{Y}-\int\frac{x^6dx}{X}$$

$$= \text{Const.} + \frac{xy}{15\gamma^3}\left(15\alpha\alpha+30\alpha\delta xy+5(4\delta\delta+2\alpha\zeta-\gamma\gamma)xxyy+15\delta\zeta x^3y^3+3\zeta\zeta x^4y^4\right).$$

19. Nunc autem formulis nostris irrationalibus X et Y eiusmodi formas inducamus, quae facilius ad quosvis casus accommodari queant, sitque

$$X=\sqrt{p(A+Cxx+Ex^4)} \quad \text{et} \quad Y=\sqrt{p(A+Cyy+Ey^4)};$$

necesse ergo est sit

$$Ap=-\alpha\gamma, \quad Ep=-\gamma\zeta, \quad Cp=\delta\delta-\gamma\gamma-\alpha\zeta,$$

unde fit

$$\alpha=\frac{-Ap}{\gamma}, \quad \zeta=\frac{-Ep}{\gamma} \quad \text{et} \quad \delta=\sqrt{\left(\gamma\gamma+Cp+\frac{AEpp}{\gamma\gamma}\right)}.$$

20. Sit iam $\gamma\gamma = A$ et $p = kk$ sumaturque $\gamma = -\sqrt{A}$ ac fiet

$$\alpha = kk\sqrt{A}, \quad \gamma = -\sqrt{A}, \quad \zeta = \frac{Ekk}{\sqrt{A}} \quad \text{et} \quad \delta = \sqrt{(A + Ckk + Ek^4)}$$

sicque erit

$$X = k\sqrt{(A + Cxx + Ex^4)} \quad \text{et} \quad Y = k\sqrt{(A + Cyy + Ey^4)}$$

et aequatio canonica prodibit

$$0 = Akk - A(xx + yy) + 2xy\sqrt{A(A + Ckk + Ek^4)} + Ekkxxyy.$$

21. Ex hac autem variabiles x et y ita a se invicem pendent, ut sit

$$X = -y\sqrt{A} + x\sqrt{(A + Ckk + Ek^4)} + \frac{Ekk}{\sqrt{A}}xxy,$$

$$Y = \quad x\sqrt{A} - y\sqrt{(A + Ckk + Ek^4)} - \frac{Ekk}{\sqrt{A}}xyy,$$

unde fit

$$y = \frac{x\sqrt{A(A + Ckk + Ek^4)} - k\sqrt{A(A + Cxx + Ex^4)}}{A - Ekkxx},$$

$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy}.$$

22. Hi igitur valores satisfacient huic aequationi integrali latissime patenti ex § 17 deductae, dum ea per $-k$ multiplicatur,

$$\int\frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}} - \int\frac{dy(\mathfrak{A} + \mathfrak{B}yy + \mathfrak{C}y^4)}{\sqrt{(A + Cyy + Ey^4)}}$$

$$= \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{3A\sqrt{A}}\left(3Akk + 3xy\sqrt{A(A + Ckk + Ek^4)} + Ekkxxyy\right)$$

$$= \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{6A\sqrt{A}}\left(3Akk + 3A(xx + yy) - Ekkxxyy\right).$$

23. Quodsi ergo curva quaepiam ita fuerit comparata, ut abscissae x respondeat arcus

$$= \int\frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}}$$

isque notetur per $\Pi. x$ et arcus alii abscissae y respondens in eadem curva

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4)}{\sqrt{(A+Cyy+Ey^4)}}$$

per $\Pi. y$, inter hos duos arcus ista relatio locum habebit

$$\Pi. x - \Pi. y = \text{Const.} + \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy}{6A\sqrt{A}}\big(3Akk + 3A(xx+yy) - Ekkxxyy\big),$$

siquidem abscissae x et y ita a se invicem pendeant, ut sit

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Cyy+Ey^4)}}{A-Ekkyy}$$

et

$$y = \frac{x\sqrt{A(A+Ckk+Ek^4)} - k\sqrt{A(A+Cxx+Ex^4)}}{A-Ekkxx}.$$

24. Ad istam autem constantem, quam aequatio integralis continet, determinandam consideretur casus, quo $y=0$ et quo fit $x=k$; quodsi iam arcus abscissae evanescenti conveniens quoque evanescat, erit pro hoc casu $\Pi. k = \text{Const.}$, quo valore substituto habebitur

$$\Pi. x - \Pi. y - \Pi. k = \frac{\mathfrak{B}kxy}{\sqrt{A}} + \frac{\mathfrak{C}kxy(kk+xx+yy)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^3x^3y^3}{6A\sqrt{A}}.$$

Hoc ergo modo terni arcus in ista curva dantur, quorum unus summam binorum reliquorum superat quantitate geometrice assignabili.

25. Hinc iam in genere patet, si curva ita fuerit comparata, ut arcus abscissae x respondens sit

$$\Pi. x = \int \frac{\mathfrak{A}dx}{\sqrt{(A+Cxx+Ex^4)}}$$

ideoque sit $\mathfrak{B}=0$ et $\mathfrak{C}=0$, tum arcuum illorum differentiam in nihilum abire; hocque ergo casu in hac curva arcuum comparatio perinde institui poterit atque in circulo. Sin autem in numeratore adsit terminus $\mathfrak{B}xx$ vel $\mathfrak{C}x^4$ vel uterque, tum arcuum illorum ternorum differentia geometrice assignabilis est ideoque arcuum comparatio perinde succedet atque in parabola. Ipsa autem comparatio eodem modo perficietur, quem in specimine priore pro circulo ac parabola exposui.

26. Quoniam terni arcus in computum veniunt, quorum abscissae sunt x, y et k, patet, quemadmodum y pendet ab x et k, eodem modo k ab x et y pendere, unde datis binis tertia ex his aequationibus determinabitur

$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy},$$

$$y = \frac{x\sqrt{A(A + Ckk + Ek^4)} - k\sqrt{A(A + Cxx + Ex^4)}}{A - Ekkxx},$$

$$k = \frac{x\sqrt{A(A + Cyy + Ey^4)} - y\sqrt{A(A + Cxx + Ex^4)}}{A - Exxyy}.$$

27. Si hinc aequatio formetur ab irrationalitate omni inmunis, prodibit

$$\begin{aligned} EEk^4x^4y^4 = &AA(2kkxx + 2kkyy + 2xxyy - k^4 - x^4 - y^4) \\ &+ 4ACkkxxyy + 2AEkkxxyy(kk + xx + yy). \end{aligned}$$

In qua cum ternae abscissae k, x, y pari modo sint immixtae, considerari poterunt earum quadrata kk, xx, yy tanquam radices huiusmodi aequationis cubicae

$$Z^3 - pZZ + qZ - r = 0,$$

et cum sit

$$p = kk + xx + yy,$$
$$q = kkxx + kkyy + xxyy,$$
$$r = kkxxyy,$$

erit

$$EErr = AA(4q - pp) + 4ACr + 2AEpr$$

sive

$$(Ap - Er)^2 = 4AAq + 4ACr.$$

28. Hac ergo inter coefficientes p, q et r relatione constituta si pro kk, xx et yy capiantur ternae radices huius aequationis cubicae

$$Z^3 - pZZ + qZ - r = 0,$$

erit pro comparatione arcuum curvae, quam (§ 23) sumus contemplati,

$$\Pi.\,x - \Pi.\,y - \Pi.\,k = \frac{\mathfrak{B}\sqrt{r}}{\sqrt{A}} + \frac{\mathfrak{C}p\sqrt{r}}{2\sqrt{A}} - \frac{\mathfrak{C}Er\sqrt{r}}{6A\sqrt{A}}.$$

29. Sint ipsae abscissae suis signis affectae $+x$, $-y$, $-k$ radices huius aequationis cubicae

$$z^3 + szz + tz - u = 0;$$

erit

$$\sqrt{r} = u, \quad q = tt + 2su \quad \text{et} \quad p = ss - 2t$$

atque

$$(Ass - 2At - Euu)^2 = 4AAtt + 8AAsu + 4ACuu$$

sive

$$t = \frac{Ass - Euu}{4A} - \frac{2Asu + Cuu}{Ass - Euu}.$$

Radices autem huius aequationis ope trisectionis anguli ita reperientur, ut sumto $v = \frac{2}{3}\sqrt{(ss - 3t)}$ et angulo Φ, cuius sit cosinus scilicet

$$\cos. \Phi = \frac{27u + 9st - 2s^3}{2(ss - 3t)\sqrt{(ss - 3t)}}\text{ [1]},$$

ipsae radices futurae sint

$$x = v \cos. \frac{1}{3}\Phi - \frac{1}{3}s, \qquad y = v \cos. \left(60^0 + \frac{1}{3}\Phi\right) - \frac{1}{3}s,$$

$$k = v \cos. \left(60^0 - \frac{1}{3}\Phi\right) - \frac{1}{3}s.$$

30. Sed relictis his, quae ad radices spectant, usum formulae inventae accuratius perpendamus ac primo quidem notatu maxime digna occurrit haec aequatio differentialis

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} = \frac{dy}{\sqrt{(A + Cyy + Ey^4)}},$$

quippe cui novimus convenire hanc aequationem integralem

$$x = \frac{y\sqrt{A(A + Ckk + Ek^4)} + k\sqrt{A(A + Cyy + Ey^4)}}{A - Ekkyy};$$

quae cum constantem novam k involvat ab arbitrio nostro pendentem, erit revera integralis completa.

1) Editio princeps: $\cos. \Phi = \frac{81u + 36st - 8s^3}{8(ss - 3t)\sqrt{(ss + 3t)}}$. Correxit A. K.

31. Si pro hoc casu ponamus

$$\int \frac{dx}{\sqrt{(A+Cxx+Ex^4)}} = \Pi.x,$$

quia posito $y=0$ fit $x=k$, erit $\Pi.x = \Pi.k + \Pi.y$. Hinc, si fiat $k=y$, ut sit

$$x = \frac{2y\sqrt{A(A+Cyy+Ey^4)}}{A-Ey^4},$$

erit $\Pi.x = 2\Pi.y$ ideoque iste valor ipsius x satisfacit huic aequationi differentiali

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} = \frac{2dy}{\sqrt{(A+Cyy+Ey^4)}};$$

quia autem novam constantem non complectitur, erit is tantum integrale incompletum.

32. Interim tamen et huius aequationis differentialis facile integrale completum exhiberi poterit. Ponatur enim

$$\frac{dy}{\sqrt{(A+Cyy+Ey^4)}} = \frac{dz}{\sqrt{(A+Czz+Ez^4)}}$$

eritque

$$y = \frac{z\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Czz+Ez^4)}}{A-Ekkzz},$$

qui valor loco y substituatur in formula

$$x = \frac{2y\sqrt{A(A+Cyy+Ey^4)}}{A-Ey^4},$$

sicque exprimetur x per z et novam constantem arbitrariam k, qui valor erit integrale completum huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} = \frac{2dz}{\sqrt{(A+Czz+Ez^4)}}.$$

33. Statuamus $\Pi.k = n\Pi.y$ ac sumamus valorem ipsius k iam esse inventum atque ex praecedentibus colligimus, si capiatur

$$x = \frac{y\sqrt{A(A+Ckk+Ek^4)} + k\sqrt{A(A+Cyy+Ey^4)}}{A-Ekkyy},$$

fore $\Pi . x = (n+1)\Pi . y$. Cum igitur casu $n=1$ sit $k=y$, valor hinc pro x inventus dabit valorem ipsius k pro casu $n=2$, unde reperitur x, ut sit $\Pi . x = 3\Pi . y$. Qui valor porro pro k sumtus eum praebebit valorem ipsius x, ut fiat $\Pi . x = 4\Pi . y$, sicque, quousque lubuerit, progredi licet.

34. Invento autem valore ipsius x, ut sit $\Pi . x = n\Pi . y$, erit is integrale particulare huius aequationis differentialis

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} = \frac{ndy}{\sqrt{(A+Cyy+Ey^4)}};$$

tum vero capiatur

$$z = \frac{x\sqrt{A(A+Ckk+Ek^4)}+k\sqrt{A(A+Cxx+Ex^4)}}{A-Ekkxx}$$

sicque obtinebitur valor integralis ipsius z completus pro hac aequatione differentiali

$$\frac{dz}{\sqrt{(A+Czz+Ez^4)}} = \frac{ndy}{\sqrt{(A+Cyy+Ey^4)}};$$

erit enim $\Pi . z = \Pi . k + \Pi . x = \Pi . k + n\Pi . y$.

35. Contemplemur nunc etiam in genere formulam latius patentem eamque ad lineam curvam $akfgpqrst$ (Fig. 1) transferamus, cuius haec sit indoles, ut posita abscissa quacunque $AK = x$ arcus ipsi respondens sit

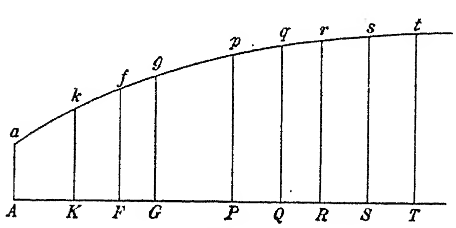

Fig. 1.

$$ak = \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{(A+Cxx+Ex^4)}},$$

quem hoc signo $\Pi . x$ indicemus. Manifestum autem est, quemadmodum haec relatio inter arcum ak et suam abscissam AK est stabilita, eandem quoque inter arcum et applicatam vel cordam aliamve rectam, ad quam arcum referre licet, constitui potuisse. Quare etsi hic x abscissam arcui ak respondentem designat, tamen quoque aliam quamvis rectam ad arcum pertinentem denotare poterit, dummodo ea evanescat ipso arcu evanescente.

36. Consideremus nunc ternas abscissas, quae sint $AK=k$, $AF=f$ et $AG=g$, quae ita a se invicem pendeant, ut sit

$$g=\frac{f\sqrt{A(A+Ckk+Ek^4)}+k\sqrt{A(A+Cff+Ef^4)}}{A-Ekkff},$$

$$f=\frac{g\sqrt{A(A+Ckk+Ek^4)}-k\sqrt{A(A+Cgg+Eg^4)}}{A-Ekkgg},$$

$$k=\frac{g\sqrt{A(A+Cff+Ef^4)}-f\sqrt{A(A+Cgg+Eg^4)}}{A-Effgg},$$

atque inter arcus $ak=\Pi.k$, $af=\Pi.f$ et $ag=\Pi.g$ haec relatio locum habebit, ut sit

$$\Pi.g-\Pi.f-\Pi.k=\text{Arc.}\,ag-\text{Arc.}\,af-\text{Arc.}\,ak=\text{Arc.}\,fg-\text{Arc.}\,ak$$
$$=\frac{\mathfrak{B}kfg}{\sqrt{A}}+\frac{\mathfrak{C}kfg(kk+ff+gg)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^3f^3g^3}{6A\sqrt{A}}.$$

37. Dato ergo quocunque arcu ak a curvae initio a sumto a quovis puncto f abscindi poterit arcus fg, ita ut differentia arcuum fg et ak geometrice assignari queat. Ob puncta enim k et f data dabuntur abscissae k et f, ex quibus per formulam primam definitur abscissa g. Vel etiam, si dentur puncta k et g, a puncto g regrediendo abscindi poterit arcus gf, qui ab arcu ak quantitate geometrica discrepet. Vel denique dato arcu quocunque fg a curvae initio a abscindi poterit arcus ak, qui ab illo quantitate geometrica discrepet.

38. Casus hic evolvi meretur, quo $f=k$; si igitur abscissa $AG=g$ (Fig. 2) ita accipiatur, ut sit

$$g=\frac{2k\sqrt{A(A+Ckk+Ek^4)}}{A-Ek^4}$$

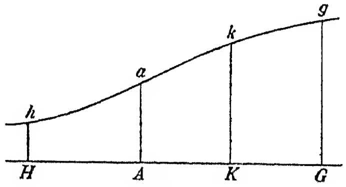

Fig. 2.

existente $AK=k$, erit

$$\text{Arc.}\,ag-2\,\text{Arc.}\,ak=\frac{\mathfrak{B}kkg}{\sqrt{A}}+\frac{\mathfrak{C}kkg(2kk+gg)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^6g^3}{6A\sqrt{A}}.$$

Quodsi iam fuerit $Ek^4>A$, valor ipsius g prodibit negativus, qui ergo retro

sumtus fit $AH = h$, ita ut sit $g = -h$ et $\Pi. g = -\Pi. h$ existente

$$h = \frac{2k\sqrt{A(A + Ckk + Ek^4)}}{Ek^4 - A},$$

eritque mutatis signis

$$\text{Arc. } ah + 2\text{Arc. } ak = \frac{\mathfrak{B}kkh}{\sqrt{A}} + \frac{\mathfrak{C}kkh(2kk + hh)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^6h^3}{6A\sqrt{A}}.$$

39. Hinc intelligitur abscissam k eiusmodi valorem obtinere posse, ut fiat $h = k$; quare si curva ex puncto a utrinque per ramos similes et aequales extendatur fueritque $AH = AK$, erit quoque Arc. $ah =$ Arc. ak; unde si sit $h = k$ seu

$$Ek^4 - A = 2\sqrt{A(A + Ckk + Ek^4)}$$

vel

$$EEk^8 - 6AEk^4 - 4ACkk - 3AA = 0,$$

erit

$$3 \text{ Arc. } ak = \frac{\mathfrak{B}k^3}{\sqrt{A}} + \frac{3\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{6A\sqrt{A}};$$

arcus ergo huic abscissae $AK = k$ respondens absolute erit rectificabilis, cum sit

$$\text{Arc. } ak = \frac{\mathfrak{B}k^3}{3\sqrt{A}} + \frac{\mathfrak{C}k^5}{2\sqrt{A}} - \frac{\mathfrak{C}Ek^9}{18A\sqrt{A}}.$$

40. Aequatio autem illa, etsi est octavi gradus, commode resolvi potest; positis enim eius factoribus

$$(k^4 + \alpha kk + \beta)(k^4 - \alpha kk + \gamma) = 0$$

reperitur

$$\beta + \gamma = \alpha\alpha - \frac{6A}{E}, \quad \beta - \gamma = \frac{4AC}{\alpha EE} \quad \text{et} \quad \beta\gamma = -\frac{3AA}{EE},$$

unde oritur

$$\alpha^4 - \frac{12A}{E}\alpha\alpha + \frac{36AA}{EE} - \frac{16AACC}{\alpha\alpha E^4} = -\frac{12AA}{EE}$$

hincque

$$\alpha\alpha = \frac{4A}{E} + \sqrt[3]{\frac{16AACC - 64A^3E}{E^4}}$$

et ob

$$\gamma = \frac{\alpha\alpha}{2} - \frac{3A}{E} - \frac{2AC}{\alpha EE}$$

erit

$$kk = \frac{1}{2}\alpha \pm \sqrt{\left(\frac{2AC}{\alpha EE} + \frac{3A}{E} - \frac{1}{4}\alpha\alpha\right)}$$

vel etiam

$$kk = -\frac{1}{2}\alpha \pm \sqrt{\left(\frac{-2AC}{\alpha EE} + \frac{3A}{E} - \frac{1}{4}\alpha\alpha\right)}.$$

41. Verum quod abscissae negativae idem arcus negative sumtus respondeat, hoc in istis curvis semper locum habet. Nam cum sit

$$\Pi.\, x = \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

si abscissa x capiatur negativa, erit

$$\Pi.\, (-x) = \int \frac{-\,dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}} = -\,\Pi.\, x.$$

Videtur ergo, quoties abscissae k in § praec. definitae respondet arcus realis, tum eiusdem arcus longitudinem geometrice assignari posse.

42. Affirmare autem non ausim hoc ratiocinium, quo arcum absolute rectificabilem elicui, semper tuto adhiberi posse; videntur enim casus existere, quibus id locum non sit habiturum. Si enim sit $\mathfrak{B} = 0$ et $\mathfrak{C} = 0$ ideoque

$$\Pi.\, x = \int \frac{\mathfrak{A}dx}{\sqrt{(A + Cxx + Ex^4)}},$$

prodiret in § 39 utique 3 Arc. $ak = 0$, cum tamen ex aequatione octavi gradus ibi exhibita non fiat abscissa $k = 0$. Verum recordandum est hanc aequationem natam esse ex hac

$$k = \frac{2k\sqrt{A(A + Ckk + Ek^4)}}{Ek^4 - A};$$

quae cum statim praebeat radicem $k = 0$, haec unica erit, quae hoc casu quaesito satisfacit reliquis existentibus omnibus ineptis.

43. Neque tamen his casibus ratiocinium omnino fallere censendum est, etiamsi pro k alia quaecunque radix accipiatur, sed potius eidem abscissae plures arcus respondere sunt putandi, quorum unus tantum isque negativus

satisfaciat; hocque ergo casu, tametsi in § 38 statuatur $h=k$, tamen inde non sequitur esse Arc. $ah=$ Arc. ak ideoque Arc. $ah+2$ Arc. $ak=3$ Arc. ak, cum eidem abscissae $h=k$ etiam alii arcus praeter Arc. ak conveniant, inter quos unus sit, qui reddat Arc. $ah+2$ Arc. $ak=0$.

44. Quod quo clarius perspiciatur, ponamus $A=1$, $C=2$ et $E=1$ existente $\mathfrak{B}=0$ et $\mathfrak{C}=0$ eritque $\Pi.\,x=\mathfrak{A}$ Arc. tang. x et Arc. $ak=\mathfrak{A}$ A tang. k atque Arc. $ah=\mathfrak{A}$ A tang. h; posito ergo

$$h=\frac{2k\sqrt{(1+2kk+k^4)}}{k^4-1}=\frac{2k}{kk-1}$$

erit $\mathfrak{A}$ A tang. $h+2\mathfrak{A}$ A tang. $k=0$. Quodsi iam ponatur $h=k$, fiet $kk=3$ et $k=\sqrt{3}$ reperieturque $\mathfrak{A}(\text{A tang.}\sqrt{3}+2\,\text{A tang.}\sqrt{3})=0$. Etsi autem est A tang. $\sqrt{3}=$ Arc. 60^0, tamen inde non sequitur $3\mathfrak{A}$ Arc. $60^0=0$, quod utique esset falsum; sed quoniam tangenti $\sqrt{3}$ convenit quoque arcus -120^0, hic valor priori loco pro A tang. $\sqrt{3}$ scriptus veritatem praebebit, scilicet

$$\mathfrak{A}(-\text{Arc.}\,120^0+2\,\text{Arc.}\,60^0)=0.$$

45. Haec igitur ambiguitas, qua eidem quantitati k, quam hic tanquam abscissam assumimus, plures valores Arc. ak respondere possunt, in causa est, quod, etiamsi in § 38 ponatur $h=k$, non tamen pro Arc. $ah+2$ Arc. ak scribere liceat 3 Arc. ak. Interim tamen nihilominus erit etiam hoc casu

$$\text{Arc.}\,ah+2\,\text{Arc.}\,ak=\frac{\mathfrak{B}k^3}{\sqrt{A}}+\frac{3\mathfrak{C}k^5}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^9}{6A\sqrt{A}};$$

abscissae enim h, etsi est $=k$, tamen praeter arcum ak alius quoque arcus conveniet, qui loco Arc. ah substitutus aequationi satisfacit. Hanc ergo ambiguitatem sedulo dispici oportet, ne in errorem inducamur.

46. Quoties autem huiusmodi ambiguitas non habet locum, ita ut eidem abscissae unicus arcus respondeat, tum sine haesitatione posita abscissa $h=k$ etiam pro Arc. ah scribere licebit Arc. ak et 3 Arc. ak pro Arc. $ah+2$ Arc. ak neque hinc ullus error erit extimescendus, quaecunque radix aequationis octavi gradus § 39 inventae pro k capiatur. Id quod evidens erit in casu, quo $\mathfrak{A}=A$, $\mathfrak{B}=2C$ et $\mathfrak{C}=3E$, quippe quo fit

$$\Pi.\,x=x\sqrt{(A+Cxx+Ex^4)}$$

ideoque quantitas algebraica et

$$\Pi.g - \Pi.f - \Pi.k = \frac{2Ckfg}{\sqrt{A}} + \frac{3Ekfg(kk+ff+gg)}{2\sqrt{A}} - \frac{EEk^3f^3g^3}{2A\sqrt{A}}.$$

47. Quodsi iam ponatur $f = k$, erit

$$g = \frac{2k\sqrt{A(A+Ckk+Ek^4)}}{A-Ek^4}$$

et

$$\sqrt{A(A+Cgg+Eg^4)} = \frac{A(gg-2kk)+Ek^4gg}{2kk}.$$

Sit nunc $g = -k$ seu

$$Ek^4 - A = 2\sqrt{A(A+Ckk+Ek^4)};$$

erit

$$\sqrt{A(A+Cgg+Eg^4)} = \frac{-A+Ek^4}{2} = \sqrt{A(A+Ckk+Ek^4)};$$

unde $\Pi.g = -\Pi.k$ et

$$-3\Pi.k = \frac{-2Ck^3}{\sqrt{A}} - \frac{9Ek^5}{2\sqrt{A}} + \frac{EEk^9}{2A\sqrt{A}} \quad \text{seu} \quad 3\Pi.k = \frac{k(4ACkk+9AEk^4-EEk^8)}{2A\sqrt{A}}.$$

At est

$$EEk^8 = 6AEk^4 + 4ACkk + 3AA,$$

unde fit

$$3\Pi.k = \frac{k(3AEk^4-3AA)}{2A\sqrt{A}} = \frac{3k(Ek^4-A)}{2\sqrt{A}} = 3k\sqrt{(A+Ckk+Ek^4)},$$

quod ob $\Pi.k = k\sqrt{(A+Ckk+Ek^4)}$ est veritati consentaneum.

48. Quanquam autem haec curva per se est rectificabilis, tamen evidenter probat id, quod volumus, scilicet contineri in nostris formulis etiam curvas irrectificabiles, in quibus modo ante exposito arcum absolute rectificabilem assignari liceat. Invento autem uno arcu rectificabili velut ak ex eo statim infiniti alii eiusdem indolis exhiberi poterunt; cum enim a quovis puncto f abscindi queat arcus fg, cuius ab illo differentia est geometrica, etiam hic arcus erit rectificabilis. Praeterea vero ex eodem arcu adhuc alii infiniti pariter rectificabiles reperientur modo sequenti, quem in genere exponere convenit.

49. Quo nostras formulas simpliciores reddamus, ponamus brevitatis gratia

$$\sqrt{A(A+Ckk+Ek^4)}=K,\quad \sqrt{A(A+Cff+Ef^4)}=F,\quad \sqrt{A(A+Cgg+Eg^4)}=G,$$

ut sit per § 36

$$g=\frac{fK+kF}{A-Ekkff},\quad f=\frac{gK-kG}{A-Ekkgg},\quad k=\frac{gF-fG}{A-Effgg}.$$

Quodsi iam fuerit

$$\Pi.x=\int\frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4)}{\sqrt{(A+Cxx+Ex^4)}},$$

erit

$$\Pi.g-\Pi.f-\Pi.k=\text{Arc.}\,ag-\text{Arc.}\,af-\text{Arc.}\,ak=\text{Arc.}\,fg-\text{Arc.}\,ak$$
$$=\frac{\mathfrak{B}kfg}{\sqrt{A}}+\frac{\mathfrak{C}kfg(kk+ff+gg)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^3f^3g^3}{6A\sqrt{A}}.$$

50. Sumantur simili modo praeter abscissam $AK=k$ aliae duae abscissae $AP=p$, $AQ=q$ positoque pariter

$$\sqrt{A(A+Cpp+Ep^4)}=P\quad\text{et}\quad\sqrt{A(A+Cqq+Eq^4)}=Q$$

ac relatione hac constituta

$$q=\frac{pK+kP}{A-Ekkpp},\quad p=\frac{qK-kQ}{A-Ekkqq},\quad k=\frac{qP-pQ}{A-Eppqq}$$

erit pro eadem curva

$$\text{Arc.}\,pq-\text{Arc.}\,ak=\frac{\mathfrak{B}kpq}{\sqrt{A}}+\frac{\mathfrak{C}kpq(kk+pp+qq)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^3p^3q^3}{6A\sqrt{A}}.$$

51. Subtracta ergo illa aequatione ab hac relinquetur

$$\text{Arc.}\,pq-\text{Arc.}\,fg$$
$$=\frac{\mathfrak{B}k(pq-fg)}{\sqrt{A}}+\frac{\mathfrak{C}kpq(kk+pp+qq)-\mathfrak{C}kfg(kk+ff+gg)}{2\sqrt{A}}-\frac{\mathfrak{C}Ek^3(p^3q^3-f^3g^3)}{6A\sqrt{A}},$$

ubi abscissae f, g, p et q ita a se invicem pendent, ut sit

$$k=\frac{gF-fG}{A-Effgg}=\frac{qP-pQ}{A-Eppqq}\quad\text{vel}\quad\frac{1}{k}=\frac{gF+fG}{A(gg-ff)}=\frac{qP+pQ}{A(qq-pp)},$$

unde simul abscissa k eliminari et relatio inter f, g, p, q definiri poterit.

52. Quo haec eliminatio facilius absolvatur, notandum est esse quoque

$$K = \frac{A(ff+gg-kk)-Ekkffgg}{2fg} = \frac{A(pp+qq-kk)-Ekkppqq}{2pq},$$

unde fit

$$kk = \frac{Apq(ff+gg)-Afg(pp+qq)}{(pq-fg)(A-Efgpq)} = \frac{(gF-fG)^2}{(A-Effgg)^2} = \frac{(qP-pQ)^2}{(A-Eppqq)^2}.$$

Erit ergo

$$pq(kk+pp+qq)-fg(kk+ff+gg) = pq(pp+qq)-fg(ff+gg) + \frac{Apq(ff+gg)-Afg(pp+qq)}{A-Efgpq}$$

hincque obtinetur

$$\text{Arc.}\, pq - \text{Arc.}\, fg = \frac{\mathfrak{B}k(pq-fg)}{\sqrt{A}} + \frac{\mathfrak{C}k(pq-fg)(ff+gg+pp+qq)}{2\sqrt{A}} - \frac{\mathfrak{C}Ek(pq-fg)^2(pq(ff+gg)-fg(pp+qq))}{6(A-Efgpq)\sqrt{A}}.$$

53. Cum igitur sit

$$kk = \frac{A(pq(ff+gg)-fg(pp+qq))}{(pq-fg)(A-Efgpq)}$$

et quatuor abscissae f, g, p, q ita a se invicem pendeant, ut sit

$$\frac{gF+fG}{gg-ff} = \frac{qP+pQ}{qq-pp},$$

patet proposito arcu quocunque fg in curva assumta semper ab alio dato puncto p abscindi posse arcum pq, qui ab illo arcu differat quantitate algebraice assignabili.

54. Quodsi porro a puncto q ulterius progrediendo capiatur punctum r, ita ut posita abscissa $AR = r$ sit

$$\frac{gF+fG}{gg-ff} = \frac{rQ+qR}{rr-qq}$$

seu

$$\frac{pq(ff+gg)-fg(pp+qq)}{(pq-fg)(A-Efgpq)} = \frac{qr(ff+gg)-fg(qq+rr)}{(qr-fg)(A-Efgqr)} = \frac{qr(pp+qq)-pq(qq+rr)}{(qr-pq)(A-Epqqr)},$$

erit quoque Arc. qr — Arc. fg = quantitati algebraicae, quae differentia ad priorem addita dabit

$$\text{Arc.}\, pr - 2\,\text{Arc.}\, fg = \text{Quant. algebr.},$$

sicque a dato puncto p abscindi potest arcus pr, qui duplum arcum propositum fg superet quantitate algebraica.

55. Simili modo, si ulterius abscissae $AS = s$, $AT = t$ etc. ita capiantur, ut sit

$$\frac{gF+fG}{gg-ff} = \frac{sR+rS}{ss-rr} = \frac{tS+sT}{tt-ss} \text{ etc.},$$

arcus ps triplum arcus fg, arcus pt quadruplum arcus fg etc. superabit quantitate geometrice assignabili. Vicissim autem dato vel arcu pr vel ps vel pt etc. reperiri poterit a dato puncto f arcus fg, qui ab illius semissi vel triente vel quadrante deficiat quantitate geometrice assignabili.

56. Evenire etiam posset, ut, licet quantitates $\mathfrak{B}$ et $\mathfrak{C}$ non sint nihilo aequales, tamen differentiae istae geometrice assignabiles evanescant; quin etiam semper una abscissarum ita definiri potest, ut haec differentia re vera in nihilum abeat. His igitur casibus in proposita curva eiusmodi bini arcus assignari poterunt, qui inter se vel aequales sint futuri vel datam rationem numeri ad numerum habituri.

57. Cum haec latissime pateant atque ad omnes curvas accommodari queant, quarum arcus pro abscissa vel alia quacunque recta variabili x ita exprimitur, ut sit

$$= \int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

conveniet istas affectiones pro nonnullis curvis determinatis evolvi, quo usus huius methodi clarius perspiciatur. Primum igitur potissimum hanc arcuum comparationem in ellipsi exponere visum est.

DE COMPARATIONE ARCUUM IN ELLIPSI

58.[1]) Sit igitur propositus quadrans ellipticus ABa (Fig. 3, p. 173), cuius centrum in A; ponatur alter semiaxis, super quo abscissae capiuntur,

1) In editione principe paragraphorum numeri abhinc desunt. A. K.

$AB = a$, alter vero $Aa = na$. Sumta ergo abscissa quacunque $AP = x$ erit applicata

$$PM = n\sqrt{(aa - xx)}$$

eiusque differentiale

$$= -\frac{nx\,dx}{\sqrt{(aa - xx)}},$$

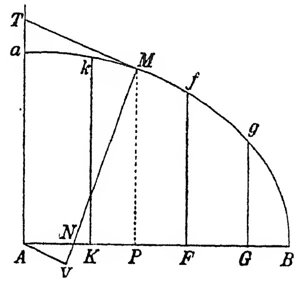

Fig. 3.

unde fit arcus huic abscissae respondens

$$aM = \int dx \sqrt{\frac{aa + (nn - 1)xx}{aa - xx}}.$$

Statuatur $1 - nn = m$, ut sit

$$aM = \int dx \sqrt{\frac{aa - mxx}{aa - xx}}.$$

Quia perinde est, uter semiaxium sit maior vel minor, sumamus AB esse maiorem; ideoque $n < 1$ et m numerus positivus unitate minor, et cum focus existat in semiaxe AB, erit eius a centro A distantia

$$= \sqrt{(aa - nnaa)} = a\sqrt{m};$$

unde valor numeri m facilius intelligitur.

Quodsi ergo arcus abscissae cuicunque $AP = x$ respondens designetur $aM = \Pi.\,x$, erit

$$\Pi.\,x = \int dx \sqrt{\frac{aa - mxx}{aa - xx}},$$

quae expressio ad formam nostram generalem reducta abibit in hanc

$$\Pi.\,x = \int \frac{dx(aa - mxx)}{\sqrt{(a^4 - (m+1)aaxx + mx^4)}}.$$

Sicque pro hoc casu habebimus istos valores

$$A = a^4, \quad C = -(m+1)aa, \quad E = m, \quad \mathfrak{A} = aa, \quad \mathfrak{B} = -m \quad \text{et} \quad \mathfrak{C} = 0.$$

Sumtis ergo tribus abscissis k, x, y, quibus respondeant arcus $\Pi.\,k$, $\Pi.\,x$, $\Pi.\,y$, ita ut sit

$$x = \frac{aay\sqrt{(a^4 - (m+1)aakk + mk^4)} + aak\sqrt{(a^4 - (m+1)aayy + my^4)}}{a^4 - mkkyy},$$

$$y = \frac{aax\sqrt{(a^4 - (m+1)aakk + mk^4)} - aak\sqrt{(a^4 - (m+1)aaxx + mx^4)}}{a^4 - mkkxx},$$

$$k = \frac{aax\sqrt{(a^4 - (m+1)aayy + my^4)} - aay\sqrt{(a^4 - (m+1)aaxx + mx^4)}}{a^4 - mxxyy},$$

hi tres arcus a se invicem ita pendebunt, ut sit

$$\Pi.\, x - \Pi.\, y - \Pi.\, k = -\frac{mkxy}{aa}.$$

His igitur praemissis sequentia problemata resolvamus.

PROBLEMA 1

59. *Proposito ellipseos arcu quocunque ak* (Fig. 3, p. 173) *ab alio quovis puncto f abscindere arcum fg, ita ut differentia arcuum ak et fg geometrice assignari queat.*

SOLUTIO

Ductis ex punctis k, f, g applicatis kK, fF, gG vocentur abscissae $AK = k$, $AF = f$, $AG = g$, quarum illae dantur, haec vero quaeritur, eruntque arcus

$$ak = \Pi.\, k, \quad af = \Pi.\, f, \quad ag = \Pi.\, g.$$

Ponatur porro brevitatis gratia secundum § 49

$$aa\sqrt{(a^4 - (m+1)aakk + mk^4)} = K,$$

$$aa\sqrt{(a^4 - (m+1)aaff + mf^4)} = F,$$

$$aa\sqrt{(a^4 - (m+1)aagg + mg^4)} = G$$

ac statuatur inter ternas abscissas ista relatio

$$g = \frac{fK + kF}{a^4 - mkkff} \quad \text{vel} \quad f = \frac{gK - kG}{a^4 - mkkgg} \quad \text{vel} \quad k = \frac{gF - fG}{a^4 - mffgg},$$

quo facto habebitur

$$\Pi.\, g - \Pi.\, f - \Pi.\, k = \text{Arc.}\, fg - \text{Arc.}\, ak = -\frac{mkfg}{aa}.$$

Puncto g ergo ita sumto, ut sit

$$AG = g = \frac{fK + kF}{a^4 - mkkff},$$

differentia arcuum ak et fg geometrice poterit assignari. Erit enim

$$\text{Arc.}\, ak - \text{Arc.}\, fg = \frac{mkfg}{aa}.$$

Q. E. I.

COROLLARIUM 1

60. Eadem solutio locum habebit, si proposito arcu ak detur punctum g, a quo regrediendo versus a abscindi oporteat arcum gf, cuius ab illo differentia debeat esse geometrica; tum enim abscissae k et g erunt datae, ex quibus valor tertiae f reperiri poterit.

COROLLARIUM 2

61. Dato etiam arcu quocunque in ellipsi fg a vertice a abscindi poterit arcus ak, ita ut differentia arcuum ak et fg fiat geometrica. Ita cuiusque arcus fg rectificatio pendebit a rectificatione arcus cuiusdam ak in vertice ellipsis a terminato.

COROLLARIUM 3

62. Relatio inter ternas abscissas k, f, g etiam ita exhiberi potest, ut sit

$$g = \frac{a^4(-kk+ff)}{fK-kF} \quad \text{vel} \quad f = \frac{a^4(-kk+gg)}{gK+kG} \quad \text{vel} \quad k = \frac{a^4(gg-ff)}{gF+fG},$$

ex quibus cum praecendentibus comparatis elicitur

$$K = \frac{a^4(ff+gg-kk)-mkkffgg}{2fg} = aa\sqrt{(aa-kk)(aa-mkk)},$$

$$F = \frac{a^4(kk+gg-ff)-mkkffgg}{2kg} = aa\sqrt{(aa-ff)(aa-mff)},$$

$$G = \frac{-a^4(kk+ff-gg)+mkkffgg}{2kf} = aa\sqrt{(aa-gg)(aa-mgg)};$$

tum vero etiam habebitur

$$fg(gg-ff)K - kg(gg-kk)F - kf(ff-kk)G = 0.$$

COROLLARIUM 4

63. Si differentia inter arcus ak et fg omnino debeat evanescere, patet id fieri non posse, nisi sit vel $k=0$ vel $f=0$ vel $g=0$. Primo casu ipse arcus ak ideoque et arcus fg evanescit, binis reliquis casibus autem alteruter terminus arcus fg in punctum a incidit fitque arcus fg arcui ak non solum aequalis, sed etiam similis.

COROLLARIUM 5

64. Quo ista abscissarum relatio facilius ad praxin transferri queat, notasse iuvabit in genere, si ad punctum M ducatur normalis MN in eamque ex A perpendiculum demittatur AV, quod parallelum erit tangenti MT, atque ponatur $AP = x$, fore

$$PM = n\sqrt{(aa - xx)}, \quad PN = nnx, \quad AN = mx, \quad MN = n\sqrt{(aa - mxx)},$$

$$AV = \frac{mx\sqrt{(aa-xx)}}{\sqrt{(aa-mxx)}}, \quad NV = \frac{mnxx}{\sqrt{(aa-mxx)}}, \quad MV = \frac{naa}{\sqrt{(aa-mxx)}},$$

$$MT = \frac{x\sqrt{(aa-mxx)}}{\sqrt{(aa-xx)}}, \quad AT = \frac{naa}{\sqrt{(aa-xx)}} \quad \text{et} \quad AV \cdot MT = mxx.$$

COROLLARIUM 6

65. Posito ergo g pro x isti valores pro puncto g reperiuntur

$$g = \frac{a^2k\sqrt{(aa-ff)(aa-mff)} + aaf\sqrt{(aa-kk)(aa-mkk)}}{a^4 - mkkff},$$

$$\sqrt{(aa-gg)} = \frac{a^3\sqrt{(aa-kk)(aa-ff)} - akf\sqrt{(aa-mkk)(aa-mff)}}{a^4 - mkkff},$$

$$\sqrt{(aa-mgg)} = \frac{a^3\sqrt{(aa-mkk)(aa-mff)} - makf\sqrt{(aa-kk)(aa-ff)}}{a^4 - mkkff}$$

atque

$$\sqrt{(aa-gg)(aa-mgg)}$$
$$= \frac{a^4kf(2maa(kk+ff) - (m+1)(a^4+mkkff)) + aa(a^4+mkkff)\sqrt{(aa-kk)(aa-mkk)(aa-ff)(aa-mff)}}{(a^4-mkkff)^2},$$

unde porro elicitur

$$aa\sqrt{(aa-mgg)} + mkf\sqrt{(aa-gg)} = a\sqrt{(aa-mkk)(aa-mff)},$$
$$aa\sqrt{(aa-gg)} + kf\sqrt{(aa-mgg)} = a\sqrt{(aa-kk)(aa-ff)}.$$

CASUS 1

66. *Proposito ellipseos arcu ak* (Fig. 4, p. 177) *in altero vertice a terminato ltero vertice B abscindere arcum Bf, ita ut arcuum ak et Bf differentia sit rica.*

Problema ergo ad hunc casum transfertur, si punctum g in vertice B statuatur seu fiat $g=a$, et quaeri oportet punctum f seu abscissam $AF=f$. Verum ob $g=a$ erit $G=0$ ideoque habebitur

$$f=\frac{aK}{a^4-maakk}=a\sqrt{\frac{aa-kk}{aa-mkk}}$$

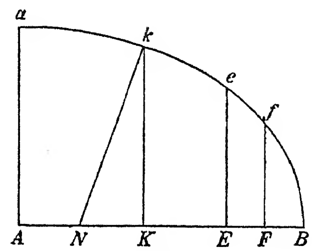

Fig. 4.

vel ducta ad punctum k normali kN capi debet

$$AF=f=\frac{AB\cdot Kk}{Nk}.$$

Hoc autem puncto ita sumto erit arcuum differentia

$$\text{Arc. } ak-\text{Arc. } Bf=\frac{mkf}{a}=mk\sqrt{\frac{aa-kk}{aa-mkk}}=\frac{AN\cdot Kk}{Nk}.$$

COROLLARIUM

67. Fieri igitur potest, ut puncta k et f in uno puncto e coeant sicque quadrans aeB in duas partes dissecetur, quarum differentia sit geometrica. Ad hoc statuatur $k=f=AE=e$ eritque

$$e=a\sqrt{\frac{aa-ee}{aa-mee}} \quad \text{seu} \quad a^4-2aaee+me^4=0,$$

unde fit

$$ee=\frac{aa\pm aa\sqrt{(1-m)}}{m}=\frac{aa(1\pm n)}{m}$$

ob $m=1-nn$. Hinc ergo erit

$$e=\frac{a}{\sqrt{(1\pm n)}}.$$

Verum quia esse debet $e<a$, erit

$$e=\frac{a}{\sqrt{(1+n)}}$$

sive

$$AE=\frac{AB^2}{\sqrt{(AB^2+AB\cdot Aa)}} \quad \text{et} \quad Ee=\frac{na\sqrt{n}}{\sqrt{(1+n)}},$$

ita ut sit

$$AE:Ee=1:n\sqrt{n}=AB\sqrt{AB}:Aa\sqrt{Aa}.$$

Hocque casu erit

$$\text{Arc. } ae-\text{Arc. } Be=a(1-n)=AB-Aa.$$

CASUS 2

68. *Proposito arcu ak (Fig. 5) in vertice a terminato ab eius altero termino k abscindere arcum kg, ita ut arcuum ak et kg differentia sit rectificabilis.*

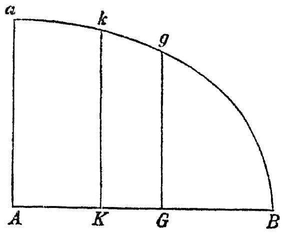

Fig. 5.

Hoc ergo casu punctum f in k incidit eritque $f = k$ hincque etiam $F = K$; unde reperitur

$$AG = g = \frac{2kK}{a^4 - mk^4} = \frac{2aak\sqrt{(aa-kk)(aa-mkk)}}{a^4 - mk^4}.$$

Sumta ergo abscissa AG huius valoris erit

$$\text{Arc. } ak - \text{Arc. } kg = \frac{mkkg}{aa} = \frac{2mk^3\sqrt{(aa-kk)(aa-mkk)}}{a^4 - mk^4}.$$

COROLLARIUM 1

69. Vicissim ergo arcus quicunque ag in vertice a terminatus ita in k in duas partes secari poterit, ut partium differentia $ak - kg$ fiat rectificabilis. Ob cognitam enim abscissam $AG = g$ abscissa quaesita $AK = k$ ex hac aequatione definiri debet

$$gg(a^4 - mk^4)^2 = 4a^4kk(aa - kk)(aa - mkk),$$

quae abit in hanc octavi gradus

$$mmggk^8 - 4ma^4k^6 - 2ma^4ggk^4 + 4(m+1)a^6k^4 - 4a^8kk + a^8gg = 0.$$

COROLLARIUM 2

70. At si huius aequationis factores ponantur

$$(mgk^4 - Akk + a^4g)(mgk^4 - Bkk + a^4g) = 0,$$

reperitur

$$A + B = \frac{4a^4}{g} \quad \text{et} \quad AB = 4(m+1)a^6 - 4ma^4gg,$$

unde

$$A - B = \frac{4aa}{g}\sqrt{(a^4 - (m+1)aagg + mg^4)},$$

ita ut sit

$$A = \frac{2a^4 + 2aa\sqrt{(aa-gg)(aa-mgg)}}{g}$$

et

$$B = \frac{2a^4 - 2aa\sqrt{(aa-gg)(aa-mgg)}}{g}.$$

Consequenter

$$k^4 = \frac{2a^4kk \pm 2aakk\sqrt{(aa-gg)(aa-mgg)} - a^4gg}{mgg}$$

et

$$kk = \frac{a^4 \pm aa\sqrt{(aa-gg)(aa-mgg)} \pm a^3\sqrt{(2aa-(m+1)gg \pm 2\sqrt{(aa-gg)(aa-mgg)})}}{mgg}.$$

COROLLARIUM 3

71. Quaternae ergo radices ipsius kk sunt

$$\text{I. } kk = \frac{a^4 + aa\sqrt{(aa-gg)(aa-mgg)} + a^3\sqrt{(aa-gg)} + a^3\sqrt{(aa-mgg)}}{mgg},$$

$$\text{II. } kk = \frac{a^4 + aa\sqrt{(aa-gg)(aa-mgg)} - a^3\sqrt{(aa-gg)} - a^3\sqrt{(aa-mgg)}}{mgg},$$

$$\text{III. } kk = \frac{a^4 - aa\sqrt{(aa-gg)(aa-mgg)} + a^3\sqrt{(aa-gg)} - a^3\sqrt{(aa-mgg)}}{mgg},$$

$$\text{IV. } kk = \frac{a^4 - aa\sqrt{(aa-gg)(aa-mgg)} - a^3\sqrt{(aa-gg)} + a^3\sqrt{(aa-mgg)}}{mgg},$$

quae adhibita ambiguitate hoc modo coniunctim repraesentari possunt

$$kk = \frac{aa}{mgg}(a \pm \sqrt{(aa-gg)})(a \pm \sqrt{(aa-mgg)}).$$

COROLLARIUM 4

72. Ipsi autem valores ipsius k erunt hinc

$$k = \pm\frac{a}{g\sqrt{m}}\left(\sqrt{\frac{a+g}{2}} \pm \sqrt{\frac{a-g}{2}}\right)\left(\sqrt{\frac{a+g\sqrt{m}}{2}} \pm \sqrt{\frac{a-g\sqrt{m}}{2}}\right),$$

qui sunt omnino numero octo, quaterni affirmativi totidemque negativi illisque aequales; manifestum autem est affirmativos tantum hic locum habere ex iisque eos, qui praebent $k < g$. Hic autem est certo

$$k = \frac{a}{g\sqrt{m}}\left(\sqrt{\frac{a+g}{2}} - \sqrt{\frac{a-g}{2}}\right)\left(\sqrt{\frac{a+g\sqrt{m}}{2}} - \sqrt{\frac{a-g\sqrt{m}}{2}}\right).$$

Nam est

$$\sqrt{\frac{a+g}{2}} + \sqrt{\frac{a-g}{2}} > \sqrt{a}, \quad \sqrt{\frac{a+g}{2}} - \sqrt{\frac{a-g}{2}} < \sqrt{g},$$

$$\sqrt{\frac{a+g\sqrt{m}}{2}} + \sqrt{\frac{a-g\sqrt{m}}{2}} > \sqrt{a}, \quad \sqrt{\frac{a+g\sqrt{m}}{2}} - \sqrt{\frac{a-g\sqrt{m}}{2}} < \sqrt{g\sqrt{m}}.$$

COROLLARIUM 5

73. Si ponatur

$$\frac{g}{a} = \cos.\eta \quad \text{et} \quad \frac{g\sqrt{m}}{a} = \cos.\theta,$$

ob $m < 1$ erit $\theta > \eta$ et formula nostra pro radicibus ipsius k inventa in hanc abibit formam

$$k = \pm \frac{a}{\cos.\theta}\left(\cos.\frac{1}{2}\eta \pm \sin.\frac{1}{2}\eta\right)\left(\cos.\frac{1}{2}\theta \pm \sin.\frac{1}{2}\theta\right)$$

seu ob

$$\cos.\theta = \cos.\frac{1}{2}\theta^2 - \sin.\frac{1}{2}\theta^2$$

habebitur

$$k = \pm a \cdot \frac{\cos.\frac{1}{2}\eta \pm \sin.\frac{1}{2}\eta}{\cos.\frac{1}{2}\theta \pm \sin.\frac{1}{2}\theta}.$$

Vel octoni valores erunt

$$k = \pm a \cdot \frac{\cos.\left(45^0 - \frac{1}{2}\eta\right)}{\cos.\left(45^0 - \frac{1}{2}\theta\right)}, \qquad k = \pm a \cdot \frac{\sin.\left(45^0 - \frac{1}{2}\eta\right)}{\cos.\left(45^0 - \frac{1}{2}\theta\right)},$$

$$k = \pm a \cdot \frac{\cos.\left(45^0 - \frac{1}{2}\eta\right)}{\sin.\left(45^0 - \frac{1}{2}\theta\right)}, \qquad k = \pm a \cdot \frac{\sin.\left(45^0 - \frac{1}{2}\eta\right)}{\sin.\left(45^0 - \frac{1}{2}\theta\right)}.$$

COROLLARIUM 6

74. Ex his valoribus secundus

$$k = a \cdot \frac{\sin.\left(45^0 - \frac{1}{2}\eta\right)}{\cos.\left(45^0 - \frac{1}{2}\theta\right)} = a \cdot \frac{\sin.\left(45^0 - \frac{1}{2}\eta\right)}{\sin.\left(45^0 + \frac{1}{2}\theta\right)}$$

semper satisfacit; fit enim, uti manifestum est, non solum $k > a$, sed etiam $k < g$ seu $k < a \cos.\eta$. Ex primo quidem valore

$$k = a \cdot \frac{\sin.\left(45^0 + \frac{1}{2}\eta\right)}{\sin.\left(45^0 + \frac{1}{2}\theta\right)}$$

semper fit $k < a$ ob $\eta < \theta$; verum ut sit $k < g$, oportet esse

$$\frac{\sin.\left(45^0+\frac{1}{2}\eta\right)}{\sin.\left(45^0+\frac{1}{2}\theta\right)} < \cos.\eta = \sin.(90^0-\eta) = 2\sin.\left(45^0-\frac{1}{2}\eta\right)\sin.\left(45^0+\frac{1}{2}\eta\right)$$

ideoque

$$1 < 2\sin.\left(45^0-\frac{1}{2}\eta\right)\sin.\left(45^0+\frac{1}{2}\theta\right)$$

seu

$$1 < \cos.\frac{1}{2}(\theta+\eta) - \cos.\left(90^0+\frac{1}{2}(\theta-\eta)\right)$$

seu

$$1 < \cos.\frac{1}{2}(\theta+\eta) + \sin.\frac{1}{2}(\theta-\eta).$$

PROBLEMA 2

75. *Proposito ellipseos arcu quocunque fg* (Fig. 6) *a dato puncto p abscindere alium arcum pq, ita ut horum arcuum differentia $fg-pq$ fiat geometrica.*

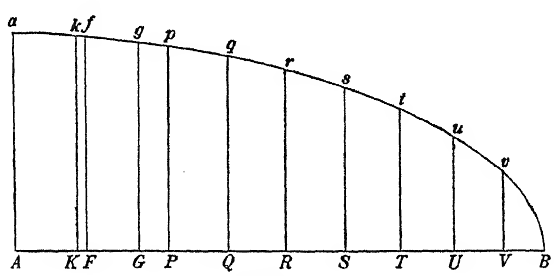

Fig. 6.

SOLUTIO

Ductis applicatis fF, gG, pP, qQ sint abscissae $AF=f$, $AG=g$, $AP=p$ et $AQ=q$, tum a vertice a capiatur arcus ak, qui datum arcum fg quantitate geometrica superet; positaque abscissa $AK=k$ ac brevitatis gratia

$$K = aa\sqrt{(aa-kk)(aa-mkk)},$$

$$F = aa\sqrt{(aa-ff)(aa-mff)},\quad G = aa\sqrt{(aa-gg)(aa-mgg)},$$

$$P = aa\sqrt{(aa-pp)(aa-mpp)} \quad \text{et} \quad Q = aa\sqrt{(aa-qq)(aa-mqq)}$$

erit primo

$$k = \frac{gF-fG}{a^4-mffgg} = \frac{a^4(gg-ff)}{gF+fG};$$

unde reperitur k, ita ut sit

$$\text{Arc. } ak - \text{Arc. } fg = \frac{mkfg}{aa}.$$

Tum vero abscissa q per problema praec. ita determinetur, ut sit

$$q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp - kk)}{pK - kP},$$

eritque

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa},$$

a qua aequatione illa subtrahatur; relinquetur

$$\text{Arc. } fg - \text{Arc. } pq = \frac{mk}{aa}(pq - fg).$$

Q. E. I.

COROLLARIUM 1

76. Cum k ab abscissis p et q pari modo pendeat atque ab f et g, erit

$$k = \frac{qP - pQ}{a^4 - mppqq} = \frac{a^4(qq - pp)}{qP + pQ}$$

ideoque abscissa q ex datis f, g et p per hanc aequationem debet definiri

$$\frac{gF - fG}{a^4 - mffgg} = \frac{qP - pQ}{a^4 - mppqq}$$

vel etiam ex hac

$$\frac{gg - ff}{gF + fG} = \frac{qq - pp}{qP + pQ};$$

atque hinc elicitur

$$q = \frac{Fgp(pp - gg) + Gfp(pp - ff) - Pfg(gg - ff)}{Ff(pp - gg) + Gg(pp - ff) - Pp(gg - ff)}.$$

COROLLARIUM 2

77. Abscissae p et q etiam ita ab abscissa k pendent, ut sit

$$aa\sqrt{(aa - mqq)} + mkp\sqrt{(aa - qq)} = a\sqrt{(aa - mkk)(aa - mpp)},$$

$$aa\sqrt{(aa - qq)} + kp\sqrt{(aa - mqq)} = a\sqrt{(aa - kk)(aa - pp)},$$

$$aa\sqrt{(aa - mpp)} - mkq\sqrt{(aa - pp)} = a\sqrt{(aa - mkk)(aa - mqq)},$$

$$aa\sqrt{(aa - pp)} - kq\sqrt{(aa - mpp)} = a\sqrt{(aa - kk)(aa - qq)},$$

$$aa\sqrt{(aa - mkk)} - mpq\sqrt{(aa - kk)} = a\sqrt{(aa - mpp)(aa - mqq)},$$

$$aa\sqrt{(aa - kk)} - pq\sqrt{(aa - mkk)} = a\sqrt{(aa - pp)(aa - qq)}.$$

COROLLARIUM 3

78. Si arcuum fg et pq differentia debeat evanescere, necesse est, ut sit vel $k=0$ vel $pq=fg$. At si $k=0$, ob

$$k=\frac{a^4(gg-ff)}{gF+fG}=\frac{a^4(qq-pp)}{qP+pQ}$$

tam arcus fg quam pq evanescit. Sin autem sit $pq=fg$, ob

$$aa\sqrt{(aa-mkk)}-mpq\sqrt{(aa-kk)}=a\sqrt{(aa-mpp)(aa-mqq)},$$

$$aa\sqrt{(aa-mkk)}-mfg\sqrt{(aa-kk)}=a\sqrt{(aa-mff)(aa-mgg)}$$

erit

$$(aa-mpp)(aa-mqq)=(aa-mff)(aa-mgg)$$

et ob

$$aa\sqrt{(aa-kk)}-pq\sqrt{(aa-mkk)}=a\sqrt{(aa-pp)(aa-qq)},$$

$$aa\sqrt{(aa-kk)}-fg\sqrt{(aa-mkk)}=a\sqrt{(aa-ff)(aa-gg)}$$

erit

$$(aa-pp)(aa-qq)=(aa-ff)(aa-gg),$$

unde patet esse vel $q=g$ et $p=f$ vel $q=f$ et $p=g$; utroque autem casu fit arcus pq non solum aequalis, sed etiam similis arcui fg.

COROLLARIUM 4

79. Si fieri posset, ut arcus pq evanesceret manente arcu fg finito, hic arcus foret rectificabilis. At evanescente arcu pq ob $q=p$ oritur $k=0$ ideoque etiam $f=g$; unde quoque arcus fg evanescit.

COROLLARIUM 5

80. Si arcus pq in altero vertice B debeat esse terminatus, ut sit $q=a$, habebimus hanc aequationem

$$a^2\sqrt{(1-m)}=\sqrt{(aa-mkk)(aa-mpp)}$$

sive

$$a^4-aakk-aapp+mkkpp=0 \quad \text{et} \quad kk=\frac{aa(aa-pp)}{aa-mpp}.$$

Qui valor substitutus in hac aequatione

$$aa\sqrt{(aa-kk)}-fg\sqrt{(aa-mkk)}=a\sqrt{(aa-ff)(aa-gg)}$$

praebet

$$0 = a^6 + 2(1-m)a^3fgp - a^4(ff+gg+pp) + maa(ffgg+ffpp+ggpp) - mffggpp;$$

unde oritur

$$p = \frac{(1-m)a^3fg \pm a\sqrt{(aa-ff)(aa-gg)(aa-mff)(aa-mgg)}}{a^4 - maaff - maagg + mffgg};$$

qui casus ad casum problematis primi redit, si modo vertices a et B inter se permutentur et loco abscissarum applicatae introducantur.

COROLLARIUM 6

81. Notari quoque meretur casus, quo punctum p in ipso puncto g assumitur, ita ut arcus pq arcui fg fiat contiguus sitque

$$\text{Arc.}\, fg - \text{Arc.}\, gq = \frac{mkg}{aa}(q-f)$$

ob $p = g$. Cum igitur sit quoque $P = G$, erit

$$\frac{gF+fG}{gg-ff} = \frac{qG+gQ}{qq-gg},$$

unde abscissa q determinatur. Vel sumta

$$k = \frac{gF-fG}{a^4-mffgg} = \frac{a^4(gg-ff)}{gF+fG}$$

erit

$$q = \frac{gK+kG}{a^4-mkkgg} = \frac{a^4(gg-kk)}{gK-kG}.$$

Hinc autem reperitur

$$q = \frac{gg}{f} - \frac{a^4(gg-ff)^2}{f} \cdot \frac{a^4-mg^4}{2FGfg + a^4(a^4(ff+gg) - 2(m+1)aaffgg - mg^4(gg-3ff))}$$

vel

$$q = \frac{2FGg(a^4-mg^4) - a^4f((a^4+mg^4)^2 - 2(m+1)aagg(a^4+mg^4) + 4ma^4g^4)}{a^4((a^4-mg^4)^2 - 4mffgg(aa-gg)(aa-mgg))}$$

vel

$$q = \frac{2FGg(a^4-mg^4) - a^4f(mg^4-2aagg+a^4)(mg^4-2maagg+a^4)}{a^4(a^4-mg^4)^2 - 4ma^4ffgg(aa-gg)(aa-mgg)}.$$

PROBLEMA 3

82. *Proposito ellipsis arcu quocunque fg a dato puncto p abscindere arcum pqr, qui a duplo illius arcus fg differat quantitate geometrice assignabili.*

SOLUTIO

Ex punctorum f et g abscissis $AF=f$, $AG=g$ earumque quantitatibus derivatis F et G quaeratur primum abscissa

$$AK=k=\frac{gF-fG}{a^4-mffgg}=\frac{a^4(gg-ff)}{gF+fG},$$

ut habeatur

$$\text{Arc. } ak-\text{Arc. } fg=\frac{mkfg}{aa}.$$

Deinde ad puncti p abscissam $AP=p$ quaeratur abscissa $AQ=q$, ut sit

$$q=\frac{pK+kP}{a^4-mkkpp}=\frac{a^4(pp-kk)}{pK-kP}$$

denotantibus litteris maiusculis K et P semper eiusmodi functiones minuscularum k et p, ut, si minuscula fuerit x, valor maiusculae respondentis futurus sit

$$X=aa\sqrt{(aa-xx)(aa-mxx)};$$

eritque

$$\text{Arc. } ak-\text{Arc. } pq=\frac{mkpq}{aa},$$

unde obtinemus

$$\text{Arc. } fg-\text{Arc. } pq=\frac{mk}{aa}(pq-fg).$$

Simili modo, si punctum q nunc tanquam datum spectetur ex eoque quaeratur punctum r, ut sit eius abscissa

$$AR=r=\frac{qK+kQ}{a^4-mkkqq}=\frac{a^4(qq-kk)}{qK-kQ},$$

habebimus

$$\text{Arc. } fg-\text{Arc. } qr=\frac{mk}{aa}(qr-fg).$$

Quare his formulis addendis eliciemus

$$2\,\text{Arc.}\,fg - \text{Arc.}\,pqr = \frac{mk}{aa}(pq + qr - 2fg)$$

sicque a dato puncto p abscidimus arcum pr, qui a duplo arcus fg discrepat quantitate algebraica. Q. E. I.

COROLLARIUM 1

83. Cum sit

$$k = \frac{a^4(gg-ff)}{gF+fG} \quad \text{et} \quad k = \frac{a^4(qq-pp)}{qP+pQ}$$

similique modo

$$k = \frac{a^4(rr-qq)}{rQ+qR},$$

habebimus has aequationes

$$\frac{gF+fG}{gg-ff} = \frac{qP+pQ}{qq-pp} = \frac{rQ+qR}{rr-qq},$$

unde ex datis abscissis f, g et p reliquae duae abscissae q et r definiuntur.

COROLLARIUM 2

84. Si arcus fg in ipso vertice a incipiat, ut sit $f=0$, erit $k=g$, unde

$$q = \frac{pG+gP}{a^4-mggpp} = \frac{a^4(pp-gg)}{pG-gP} \quad \text{et} \quad r = \frac{qG+gQ}{a^4-mggqq} = \frac{a^4(qq-gg)}{qG-gQ}.$$

Ac si praeterea punctum p in altero vertice A detur, ut sit $p=a$ et $P=0$, erit

$$q = \frac{G}{a^3-magg} = \frac{a\sqrt{(aa-gg)(aa-mgg)}}{aa-mgg};$$

hinc

$$aa-qq = \frac{aagg(1-m)(aa-mgg)}{(aa-mgg)^2} = \frac{(1-m)aagg}{aa-mgg}$$

et

$$aa-mqq = \frac{a^4(1-m)(aa-mgg)}{(aa-mgg)^2} = \frac{(1-m)a^4}{aa-mgg}, \quad \text{unde} \quad Q = \frac{-(1-m)a^5g}{aa-mgg},$$

quia applicata in partem inferiorem cadere debet, eritque

$$r = \frac{a(a^4-2aagg+mg^4)}{a^4-2maagg+mg^4}.$$

COROLLARIUM 3

85. Hoc ergo casu sumto r (Fig. 7) in superiore quadrante, ut posita abscissa $AG = g$ sit

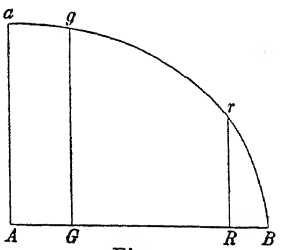

Fig. 7.

$$AR = r = \frac{a(a^4 - 2aagg + mg^4)}{a^4 - 2maagg + mg^4}$$

seu

$$BR = a - r = \frac{2(1-m)a^3gg}{a^4 - 2maagg + mg^4},$$

erit

$$2\,\text{Arc.}\,ag - \text{Arc.}\,Br = \text{Quant. algebr.} = \frac{mg}{aa}(aq + rq) = \frac{mgq}{aa}(a + r)$$

ideoque

$$2\,\text{Arc.}\,ag - \text{Arc.}\,Br = \frac{2mg(aa-gg)\sqrt{(aa-gg)(aa-mgg)}}{a^4 - 2maagg + mg^4}.$$

COROLLARIUM 4

86. Si puncta g et r in unum debeant coalescere, ut sit $r = g$, valor abscissae communis $AG = AR = g$ ex hac aequatione quinti gradus debet determinari

$$mg^5 - mag^4 - 2maag^3 + 2a^3gg + a^4g - a^5 = 0.$$

Ita, si sit $m = \frac{1}{2}$ et $a = 1$, habebitur

$$g^5 - g^4 - 2g^3 + 4gg + 2g - 2 = 0.$$

Si esset $m = \frac{4}{3+\sqrt{2}}$, prodiret $g = \frac{a}{\sqrt{2}}$ foretque

$$2\,\text{Arc.}\,ag - \text{Arc.}\,Bg = a\sqrt{\frac{2+2\sqrt{2}}{3+\sqrt{2}}}.$$

PROBLEMA 4

87. *Proposito arcu ellipseos quocunque* fg (Fig. 6, p. 181) *invenire arcum* pqr, *qui sit praecise duplo maior.*

SOLUTIO

In solutione ergo praecedentis problematis efficiendum est, ut sit

$$pq + qr - 2fg = 0,$$

eritque tum 2 Arc. $fg =$ Arc. pqr. Hic autem ob arcum fg datum in ellipsi praeter semiaxes $AB = a$ et $Aa = a\sqrt{(1-m)}$ dantur abscissae $AF = f$ et $AG = g$ cum valoribus derivatis F et G, unde quaeratur

$$k = \frac{a^4(gg - ff)}{gF + fG};$$

simulque erit eius valor derivatus

$$K = \frac{a^4(ff + gg - kk) - mkkffgg}{2fg}$$

(per coroll. 3. probl. 1). Simili autem modo abscissae p et q ab k pendent, ut sit

$$K = \frac{a^4(pp + qq - kk) - mkkppqq}{2pq},$$

itemque ex abscissis q et r erit

$$K = \frac{a^4(qq + rr - kk) - mkkqqrr}{2qr}.$$

At ex aequatione $pq + qr = 2fg$ est $q = \frac{2fg}{p+r}$, unde obtinebimus has duas aequationes

$$K = \frac{a^4(pp - kk)(p+r)^2 + 4a^4ffgg - 4mffggkkpp}{4fgp(p+r)},$$

$$K = \frac{a^4(rr - kk)(p+r)^2 + 4a^4ffgg - 4mffggkkrr}{4fgr(p+r)},$$

ex quibus ambae abscissae p et r arcum quaesitum pr determinantes definiri poterunt. Hinc ergo primum elicimus eliminando K ac per $p - r$ dividendo

$$a^4pr(p+r)^2 + a^4kk(p+r)^2 - 4a^4ffgg - 4mffggkkpr = 0.$$

Deinde addendo illas aequationes habebimus

$$2K = \frac{a^4pr(p+r)^3 - a^4kk(p+r)^3 + 4a^4ffgg(p+r) - 4mffggkkpr(p+r)}{4fgpr(p+r)}.$$

Ex illa autem est

$$a^4(p+r)^2 = \frac{4ffgg(a^4 + mkkpr)}{pr + kk},$$

qui valor in hac substitutus praebet

$$8Kfgpr = \frac{4ffgg(pr-kk)(a^4+mkkpr)}{pr+kk} + 4a^4ffgg - 4mffggkkpr$$

sive

$$\frac{2Kpr(pr+kk)}{fg} = 2a^4pr - 2mk^4pr;$$

unde elicitur

$$pr = \frac{(a^4-mk^4)fg - Kkk}{K} = \frac{ffgg(2a^4-mk^4) - a^4kk(ff+gg-kk)}{a^4(ff+gg-kk) - mffggkk}$$

et

$$(p+r)^2 = \frac{4fg}{a^4}(K+mfgkk) = \frac{2a^4(ff+gg-kk)+2mffggkk}{a^4}.$$

Ergo

$$p+r = \frac{\sqrt{2(a^4(ff+gg-kk)+mffggkk)}}{aa}.$$

Porro

$$r-p = \frac{\sqrt{2(a^8(gg-ff)^2 - a^8k^4 + 2ma^4ffggk^4 - mmf^4g^4k^4)}}{aa\sqrt{(a^4(ff+gg-kk)-mffggkk)}}$$

seu

$$r-p = \frac{\sqrt{2(a^8(gg-ff)^2 - k^4(a^4-mffgg)^2)}}{aa\sqrt{(a^4(ff+gg-kk)-mffggkk)}}.$$

Cum autem sit

$$a^4(gg-ff) = k(gF+fG) \quad \text{et} \quad a^4 - mffgg = \frac{gF-fG}{k},$$

erit

$$r-p = \frac{2k}{aa}\sqrt{\frac{FG}{K}},$$

unde ob

$$r+p = \frac{\sqrt{2(a^4(ff+gg-kk)+mffggkk)}}{aa} = \frac{2}{aa}\sqrt{fg(K+mfgkk)}$$

utraque abscissa p et r innotescit. Q. E. I.

COROLLARIUM 1

88. Cum sit

$$k = \frac{gF-fG}{a^4-mffgg}$$

et

$$K = \frac{(a^4+mffgg)FG - a^6fg(2maa(ff+gg)-(m+1)(a^4+mffgg))}{(a^4-mffgg)^2},$$

erit

$$r+p=\frac{2}{aa}\sqrt{\frac{fgFG-ma^4ffgg(ff+gg)+(m+1)a^6ffgg}{a^4-mffgg}},$$

$$r-p=\frac{2(gF-fG)}{aa}\sqrt{\frac{FG}{(a^4+mffgg)(FG+(m+1)a^6fg)-2ma^8fg(ff+gg)}}.$$

COROLLARIUM 2

89. Si arcus datus fg in vertice a terminetur, ut sit $f=0$ et $F=a^4$, prodit $p+r=0$ et $r-p=2g$, unde $p=-g$ et $r=g$; arcus ergo duplus utrinque circa a aequaliter extenditur utrumque semissem arcui fg seu ag similem habens et aequalem. Idem evenit, si arcus datus in altero vertice B terminetur, ut sit $g=a$ et $G=0$; tum enim fit $r-p=0$ et $r+p=2f$ ideoque $r=p=f$.

COROLLARIUM 3

90. Quemadmodum his casibus, ubi arcus propositus fg in altero vertice terminatur, eius arcus duplus per se est manifestus, ita, si arcus propositus in neutro vertice terminatur, assignatio arcus dupli maxime est difficilis; quippe qui arcus geometrice ne bisecari quidem potest.

COROLLARIUM 4

91. Hinc etiam patet, si detur vicissim arcus pr, inveniri posse arcum fg, qui eius exacte futurus sit semissis; sed hoc non nisi molestissimis calculis praestari poterit. At si arcus duplus pqr quadranti elliptico sit aequalis seu $p=0$ et $r=a$, non difficulter arcus assignabitur eius semissi aequalis. Primo enim erit

$$q=k \quad \text{et} \quad k=a\sqrt{\frac{1-\sqrt{(1-m)}}{m}}$$

sicque innotescit tam k quam

$$K=a^4\sqrt{\frac{1-m}{m}}\left(1-\sqrt{(1-m)}\right).$$

Porro est

$$2fg=ak \quad \text{et} \quad ff+gg=\frac{Kk}{a^3}+kk+\frac{mk^4}{4aa}.$$

At est

$$m=\frac{2aakk-a^4}{k^4} \quad \text{ideoque} \quad ff+gg=\frac{2kk+3aa}{4};$$

unde

$$g+f=\tfrac{1}{2}\sqrt{(2kk+3aa+4ak)}$$

et

$$g-f=\tfrac{1}{2}\sqrt{(2kk+3aa-4ak)}$$

ideoque

$$f=\tfrac{1}{4}\sqrt{(3aa+4ak+2kk)}-\tfrac{1}{4}\sqrt{(3aa-4ak+2kk)},$$

$$g=\tfrac{1}{4}\sqrt{(3aa+4ak+2kk)}+\tfrac{1}{4}\sqrt{(3aa-4ak+2kk)}.$$

COROLLARIUM 5

92. Si ponatur alter semiaxis $Aa=b$ existente altero $AB=a$, ut sit $m=\frac{aa-bb}{aa}$, erit pro hoc casu $k=a\sqrt{\frac{a}{a+b}}$, quo valore substituto habebitur

$$g\pm f=\frac{a}{2}\sqrt{\left(\frac{5a+3b}{a+b}\pm 4\sqrt{\frac{a}{a+b}}\right)};$$

unde fit

$$f=\frac{a}{2}\sqrt{\frac{5a+3b-\sqrt{(9aa+14ab+9bb)}}{2(a+b)}},$$

$$g=\frac{a}{2}\sqrt{\frac{5a+3b+\sqrt{(9aa+14ab+9bb)}}{2(a+b)}}$$

sicque abscissae pro utroque termino arcus fg reperiuntur, qui est semissis totius arcus quadrantis.

COROLLARIUM 6

93. Hoc ergo casu erit

$$ff+gg=\frac{aa(5a+3b)}{4(a+b)}=aa+\frac{aa(a-b)}{4(a+b)}$$

et

$$fg=\frac{aa}{2}\sqrt{\frac{a}{a+b}} \quad \text{et} \quad 2fg=aa\sqrt{\frac{a}{a+b}};$$

si exempli gratia sit $a=25$ et $b=119$, reperietur

$$f=\frac{25}{3\sqrt{2}} \quad \text{et} \quad g=\frac{125}{4\sqrt{2}}.$$

SCHOLION

94. Hinc ergo solutionem nacti sumus istius non inelegantis problematis:

Proposito ellipsis quadrante BAa (Fig. 8) *geometrice in eo abscindere arcum fg, qui praecise aequalis sit semissi totius arcus quadrantis $afgB$.*

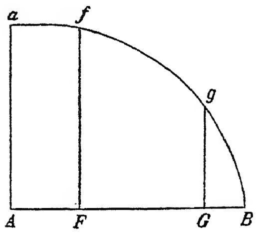

Fig. 8.

Positis enim semiaxibus $AB = a$ et $Aa = b$ pro punctis quaesitis f et g erunt abscissae

$$AF = \frac{a}{2}\sqrt{\frac{5a+3b-\sqrt{(9aa+14ab+9bb)}}{2(a+b)}},$$

$$AG = \frac{a}{2}\sqrt{\frac{5a+3b+\sqrt{(9aa+14ab+9bb)}}{2(a+b)}},$$

unde pro iisdem punctis eliciuntur applicatae

$$Ff = \frac{b}{2}\sqrt{\frac{3a+5b+\sqrt{(9aa+14ab+9bb)}}{2(a+b)}},$$

$$Gg = \frac{b}{2}\sqrt{\frac{3a+5b-\sqrt{(9aa+14ab+9bb)}}{2(a+b)}}.$$

PROBLEMA 5

95. *Datum ellipseos arcum pr* (Fig. 6, p. 181) *in duas partes secare pq et qr, ita ut differentia harum partium $pq - qr$ sit geometrice assignabilis.*

SOLUTIO

Positis ut in problemate praecedente $AP = p$, $AQ = q$ et $AR = r$ existentibus semiaxibus $AB = a$ et $Aa = a\sqrt{(1-m)}$ quaeratur a vertice a arcus ak, ut posita eius abscissa $AK = k$ sit

$$k = \frac{qP - pQ}{a^4 - mppqq} = \frac{a^4(qq-pp)}{qP+pQ},$$

eritque

$$\text{Arc. } ak - \text{Arc. } pq = \frac{mkpq}{aa}.$$

Tum vero sit etiam

$$k = \frac{rQ - qR}{a^4 - mqqrr} = \frac{a^4(rr-qq)}{rQ+qR};$$

erit quoque

$$\text{Arc. } ak - \text{Arc. } qr = \frac{mkqr}{aa}$$

ideoque

$$\text{Arc. } pq - \text{Arc. } qr = \frac{mkq}{aa}(r-p).$$

Cum igitur dentur abscissae p et r cum suis derivatis P et R, abscissa puncti quaesiti q ex hac aequatione definiri debebit

$$\frac{qP+pQ}{qq-pp} = \frac{rQ+qR}{rr-qq}$$

seu

$$Pq(rr-qq) - Rq(qq-pp) = Q(p+r)(qq-pr),$$

quae aequatio quadrata ac tum per $(qq-pp)(rr-qq)$ divisa dat

$$a^4((p+r)^2 - 2qq) - 2(m+1)aaprqq + mqq(qq(p+r)^2 - 2pprr) = 2qqPR : a^4$$

sive

$$q^4 = \frac{2qq\left(\frac{PR}{a^4} + mpprr + (m+1)aapr + a^4\right) - a^4(p+r)^2}{m(p+r)^2},$$

ex qua aequatione valor abscissae q definiri poterit. Q. E. I.

COROLLARIUM 1

96. Si totus quadrans in duas partes, quarum differentia sit geometrica, dividi debeat, poni debet $p=0$ et $r=a$; unde fit $P=a^4$ et $R=0$ indeque

$$q^4 = \frac{2aaqq - a^4}{m} \quad \text{et} \quad qq = \frac{aa(1-\sqrt{(1-m)})}{m} \quad \text{et} \quad q = a\sqrt{\frac{1-\sqrt{(1-m)}}{m}},$$

quae est eadem determinatio, quam supra iam in coroll. casus 1 probl. 1 invenimus.

COROLLARIUM 2

97. Si abscissarum p et r altera sit negativa alterique aequalis seu $p+r=0$, habebitur statim vel $q=0$ vel

$$Prr - Pqq - Rqq + Rpp = 0 \quad \text{seu} \quad qq = \frac{Prr+Rpp}{P+R} \quad \text{ideoque} \quad P+R=0.$$

Manifestum autem est, si utraque applicata Pp et Rr fuerit affirmativa, fore $R=P$ atque tum locum habere $q=0$.

PROBLEMA 6

98. *Si ellipsis $ADBFA$ (Fig. 9) per diametrum quamcunque ECF fuerit bisecta, semicircumferentiam EBF ita secare in puncto M, ut partium EM et FM differentia sit geometrice assignabilis.*

SOLUTIO

Etsi hoc problema in praecedente continetur, tamen solutio inde deduci nequit, propterea quod tam $p+r=0$ quam $P+R=0$; peculiari ergo modo solutio debet investigari. Positis ergo semiaxibus $CA=a$, $CD=b=a\sqrt{(1-m)}$ sit pro altero termino E arcus propositi abscissa $CP=p$; erit applicata $PE=\frac{b}{a}\sqrt{(aa-pp)}$, quae coordinatae negative sumtae ad alterum terminum F pertinebunt; quae autem sint r et $\frac{b}{a}\sqrt{(aa-rr)}$, ita ut sit $r=-p$ et $\sqrt{(aa-rr)}=-\sqrt{(aa-pp)}$. Cum nunc sumta quadam nova abscissa k positaque quaesita $CQ=q$ sit ex coroll. 2 probl. 2

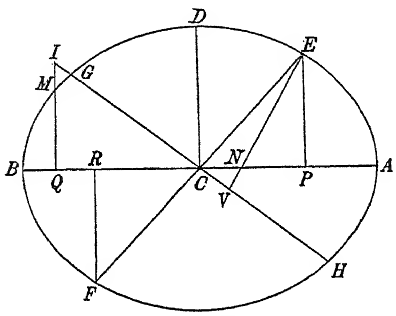

Fig. 9.

$$aa\sqrt{(aa-kk)}-pq\sqrt{(aa-mkk)}=a\sqrt{(aa-pp)(aa-qq)},$$
$$aa\sqrt{(aa-kk)}-qr\sqrt{(aa-mkk)}=a\sqrt{(aa-qq)(aa-rr)},$$

haec ultima aequatio ob

$$r=-p \quad \text{et} \quad \sqrt{(aa-rr)}=-\sqrt{(aa-pp)}$$

abit in hanc

$$aa\sqrt{(aa-kk)}+pq\sqrt{(aa-mkk)}=-a\sqrt{(aa-pp)(aa-qq)},$$

quae ad primam addita dat

$$2aa\sqrt{(aa-kk)}=0 \quad \text{ideoque} \quad k=a;$$

qui valor in altera substitutus dat

$$-pq\sqrt{(1-m)}=\sqrt{(aa-pp)(aa-qq)}$$

ideoque

$$\frac{-q}{\sqrt{(aa-qq)}}=\frac{\sqrt{(aa-pp)}}{p\sqrt{(1-m)}},$$

consequenter

$$q = -\frac{a\sqrt{(aa-pp)}}{\sqrt{(aa-mpp)}},$$

ubi signum negativum indicat q in parte abscissarum negativa capi oportere. Ducatur ad E normalis in curvam EN; erit

$$\frac{PE}{EN} = \frac{\sqrt{(aa-pp)}}{\sqrt{(aa-mpp)}}.$$

Ergo $CQ = \frac{a\cdot PE}{EN}$. Sit porro GH diameter coniugata, cui normalis EN in V occurrat; erit

$$\frac{PE}{EN} = \frac{CV}{CN} = \frac{CQ}{CI}$$

producta CG ad concursum cum applicata QM in I. Quare ob $CQ = \frac{a\cdot CQ}{CI}$ prodit $CI = a = CA$. Unde haec sequitur constructio facilis: Diameter coniugata GH ultra G in I continuetur, ut fiat $CI = CA$; ex I in axem AB demittatur perpendiculum IQ, quod ellipsin in puncto quaesito M secabit. At ob $k = a$ erit

$$\text{Arc. } EM - \text{Arc. } FM = -\frac{2mpq}{a} = \frac{2\,mp\cdot PE}{EN} = \frac{2\,CN\cdot CV}{CN} = 2\,CV$$

ob $CN = mp$. Q. E. I.

COROLLARIUM 1

99. Si ex iisdem aequationibus binis eliminando k problema praecedens generaliter solvatur, sequens obtinebitur aequatio

$$mq^4(r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)})^2-2aaqq(aa+mpr)(aa-pr-\sqrt{(aa-pp)(aa-rr)}) \\ + a^6(\sqrt{(aa-pp)}-\sqrt{(aa-rr)})^2 = 0,$$

unde per resolutionem adipiscimur

$$qq = \frac{aa(aa-pr-\sqrt{(aa-pp)(aa-rr)})(aa+mpr\pm\sqrt{(aa-mpp)(aa-mrr)})}{m(r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)})^2},$$

$$q = \frac{a\left(\sqrt{\frac{(a+r)(a-p)}{2}}-\sqrt{\frac{(a-r)(a+p)}{2}}\right)\left(\sqrt{\frac{(a+p\sqrt{m})(a+r\sqrt{m})}{2m}}\pm\sqrt{\frac{(a-p\sqrt{m})(a-r\sqrt{m})}{2m}}\right)}{r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)}}.$$

COROLLARIUM 2

100. Quanquam haec solutio re a solutione problematis praecedentis non discrepat, tamen statim solutionem praesentis suppeditat. Si enim ponamus

$$r=-p \quad \text{et} \quad \sqrt{(aa-rr)}=-\sqrt{(aa-pp)},$$

aequatio prima coroll. praec. transit in hanc formam

$$-2aaqq(aa-mpp)\cdot 2aa+a^6\big(2\sqrt{(aa-pp)}\big)^2=0$$

seu

$$qq=\frac{aa(aa-pp)}{aa-mpp}.$$

COROLLARIUM 3

101. Si ex duabus primis aequationibus eliminemus q, obtinebimus

$$q=\frac{aa\big(\sqrt{(aa-pp)}-\sqrt{(aa-rr)}\big)\sqrt{(aa-kk)}}{\big(r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)}\big)\sqrt{(aa-mkk)}}$$

et

$$\sqrt{(aa-qq)}=\frac{a(r-p)\sqrt{(aa-kk)}}{r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)}};$$

unde fit

$$\begin{aligned}a^4(aa-kk)\big(\sqrt{(aa-pp)}-\sqrt{(aa-rr)}\big)^2+a^2(aa-kk)(aa-mkk)(r-p)^2\\=aa(aa-mkk)\big(r\sqrt{(aa-pp)}-p\sqrt{(aa-rr)}\big)^2\end{aligned}$$

sive

$$\begin{aligned}mk^4(r-p)^2=2kk(aa-mpr)\big(aa-pr-\sqrt{(aa-pp)(aa-rr)}\big)\\-aa\big(aa-pr-\sqrt{(aa-pp)(aa-rr)}\big)^2,\end{aligned}$$

unde fit

$$kk=\frac{\big(aa-pr-\sqrt{(aa-pp)(aa-rr)}\big)\big(aa-mpr-\sqrt{(aa-mpp)(aa-mrr)}\big)}{m(r-p)^2},$$

hincque colligitur

$$k=\frac{\Big(\sqrt{\frac{(a+r)(a-p)}{2}}-\sqrt{\frac{(a-r)(a+p)}{2}}\Big)\Big(\sqrt{\frac{(a+r\sqrt{m})(a-p\sqrt{m})}{2m}}-\sqrt{\frac{(a-r\sqrt{m})(a+p\sqrt{m})}{2m}}\Big)}{r-p}.$$

COROLLARIUM 4

102. Hinc erit

$$kq = \frac{aa(aa - pr - \sqrt{(aa-pp)(aa-rr)})(\sqrt{(aa-mpp)} - \sqrt{(aa-mrr)})}{m(r-p)(r\sqrt{(aa-pp)} - p\sqrt{(aa-rr)})}.$$

Quare cum arcuum pq et qr differentia sit $= \frac{mkq}{aa}(r-p)$, habebimus generaliter

$$\text{Arc.}\, pq - \text{Arc.}\, qr = \frac{(aa - pr - \sqrt{(aa-pp)(aa-rr)})(\sqrt{(aa-mpp)} - \sqrt{(aa-mrr)})}{r\sqrt{(aa-pp)} - p\sqrt{(aa-rr)}},$$

siquidem punctum q ex coroll. 1 definiatur. Erit ergo

$$\text{Arc.}\, pq - \text{Arc.}\, qr = \frac{(\sqrt{(aa-pp)} - \sqrt{(aa-rr)})(\sqrt{(aa-mpp)} - \sqrt{(aa-mrr)})}{r+p}$$

et

$$q = \frac{\left(\sqrt{\frac{(a+r)(a+p)}{2}} - \sqrt{\frac{(a-r)(a-p)}{2}}\right)\left(\sqrt{\frac{(a+p\sqrt{m})(a+r\sqrt{m})}{2m}} - \sqrt{\frac{(a-p\sqrt{m})(a-r\sqrt{m})}{2m}}\right).}{p+r}.$$

PROBLEMA 7

103. *Proposito ellipsis arcu quocunque* fg (Fig. 6, p. 181) *a dato puncto* p *abscindere arcum* $pqrs$, *qui ab illius arcus* fg *triplo differat quantitate geometrice assignabili.*

SOLUTIO

Sint ut hactenus punctorum datorum f, g et p abscissae $AF = f$, $AG = g$, $AP = p$ ac quaeratur primo arcus ak, cuius abscissa sit

$$AK = k = \frac{gF - fG}{a^4 - mffgg} = \frac{a^4(gg-ff)}{gF + fG},$$

ut sit

$$\text{Arc.}\, ak - \text{Arc.}\, fg = \frac{mkfg}{aa}.$$

Tum quaeratur punctum q, ut sit

$$AQ = q = \frac{pK + kP}{a^4 - mkkpp} = \frac{a^4(pp-kk)}{pK - kP}$$

indeque

$$Q = \frac{a^4(qq-pp)-kk(a^4-mppqq)}{2kp} = \frac{pq(qq-pp)K-kq(qq-kk)P}{kp(pp-kk)},$$

eritque

$$\text{Arc.}\, fg - \text{Arc.}\, pq = \frac{mk}{aa}(pq - fg).$$

Simili modo porro quaeratur punctum r, ut sit

$$AR = r = \frac{qK+kQ}{aa-mkkqq} = \frac{a^4(qq-kk)}{qK-kQ}$$

et

$$R = \frac{a^4(rr-qq)-kk(a^4-mqqrr)}{2kq} = \frac{qr(rr-qq)K-kr(rr-kk)Q}{kq(qq-kk)},$$

et cum sit

$$\text{Arc.}\, fg - \text{Arc.}\, qr = \frac{mk}{aa}(qr - fg),$$

erit

$$2\,\text{Arc.}\, fg - \text{Arc.}\, pqr = \frac{mk}{aa}(pq + qr - 2fg).$$

Hinc pari modo definiamus punctum s, ut sit abscissa

$$AS = s = \frac{rK+kR}{a^4-mkkrr} = \frac{a^4(rr-kk)}{rK-kR}$$

et

$$S = \frac{a^4(ss-rr)-kk(a^4-mrrss)}{2kr} = \frac{rs(ss-rr)K-ks(ss-kk)R}{kr(rr-kk)},$$

et quia erit

$$\text{Arc.}\, fg - \text{Arc.}\, rs = \frac{mk}{aa}(rs - fg),$$

habebitur

$$3\,\text{Arc.}\, fg - \text{Arc.}\, pqrs = \frac{mk}{aa}(pq + qr + rs - 3fg).$$

Q. E. I.

COROLLARIUM 1

104. Simili modo progrediendo manifestum est definiri a dato puncto p posse arcum pt, qui a quadruplo arcus dati fg deficiat quantitate algebraica, atque hoc modo operationem continuari posse, quousque lubuerit.

COROLLARIUM 2

105. Si arcus datus fg toti quadranti aequetur, ut sit $f=0$ et $g=a$ ideoque $F=a^4$ et $G=0$, erit $k=a$ et $K=0$. Hinc reperitur

$$q = \frac{P}{a(aa-mpp)} = a\sqrt{\frac{aa-pp}{aa-mpp}}$$

et

$$Q = \frac{-q(qq-aa)}{p(pp-aa)}P = \frac{-(aa-qq)PP}{ap(aa-mpp)(aa-pp)} = -\frac{a^3(aa-qq)}{p};$$

at est

$$aa-qq = \frac{a(1-m)pp}{aa-mpp}, \quad \text{unde} \quad Q = \frac{-(1-m)a^5p}{aa-mpp}.$$

Porro

$$r = \frac{Q}{a(aa-mqq)} = -p$$

et

$$R = -aa\sqrt{(aa-pp)(aa-mpp)} = -P.$$

Denique erit

$$s = \frac{-P}{a(aa-mpp)} = -a\sqrt{\frac{aa-pp}{aa-mpp}} = -q \quad \text{et} \quad S = -Q = \frac{(1-m)a^5p}{aa-mpp}$$

fietque

$$3\,\text{Arc.}\,fg - \text{Arc.}\,pqrs = \frac{m}{a}pq = mp\sqrt{\frac{aa-pp}{aa-mpp}}.$$

COROLLARIUM 3

106. Punctum p quoque ita definiri poterit, ut fiat

$$pq + qr + rs = 3fg,$$

quo casu arcus $pqrs$ exacte aequabitur triplo arcus dati fg. Atque ita porro arcus inveniri poterit, qui ad arcum datum fg aliam quamvis rationem multiplicem teneat.

SCHOLION

107. Omnia haec problemata, quae hic pro ellipsi tractavi, simili modo pro hyperbola resolvi poterunt; ita etiam dato quocunque hyperbolae arcu a proposito quovis eiusdem hyperbolae puncto arcus abscindi poterit, qui discrepet vel ab illo ipso arcu vel ab eius duplo vel triplo vel ab alio quovis multiplo quantitate geometrice assignabili. Deinde etiam hoc punctum ita assumere licebit, ut differentia plane in nihilum abeat, quo casu dato quocunque hyperbolae arcu alius arcus assignari poterit, qui vel eius duplo vel triplo vel alii cuivis multiplo exacte sit aequalis. Unde perspicuum est, si proposito arcu inventus sit alius arcus, qui ad illum teneat rationem μ ad 1, similique

modo alius quaeratur arcus, qui ad eundem teneat rationem ν ad 1, tum hoc pacto duos haberi arcus hyperbolicos, qui inter se teneant rationem μ ad ν, sicque infinitis modis bini arcus exhiberi poterunt, qui sint in ratione quacunque numeri ad numerum. Neque vero huiusmodi problemata tantum pro hyperbola resolvi poterunt, sed omnino pro aliis curvis quibuscunque, quae ita sint comparata, ut arcus abscissae vel alii cuicunque lineae rectae variabili x respondens contineatur in hac formula

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4)}{\sqrt{(A + Cxx + Ex^4)}},$$

quae etiam per regulas initio datas ita latius extendi potest, ut ad hanc formam revocetur

$$\int \frac{dx(\mathfrak{A} + \mathfrak{B}xx + \mathfrak{C}x^4 + \mathfrak{D}x^6 + \mathfrak{E}x^8 + etc.)}{\sqrt{(A + Cxx + Ex^4)}};$$

sed in praesentia neque hyperbolae neque aliis huius generis curvis diutius immorandum esse arbitror.

DEMONSTRATIO THEOREMATIS ET SOLUTIO PROBLEMATIS IN ACTIS ERUD. LIPSIENSIBUS PROPOSITORUM[1]

Commentatio 264 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 7 (1758/9), 1761, p, 128—162
Summarium ibidem p. 10—11

SUMMARIUM

Cum in Actis Lips. theorema hoc ac problema sine nomine sint proposita, Cel. Auctor hic statim se eorum esse inventorem profitetur. Utrumque eximiam ellipseos proprietatem complectitur. In theoremate enim docetur, quomodo dimidia ellipsis diametro quacunque terminata ita in duas partes secanda sit, ut partium differentia geometrice assignari queat, quae ipsa divisio cum partium differentia in eo exponitur, ut a geometris demonstratio investigaretur. Prodiit quidem nuper[2]) in Actis Sociorum Academiae Parisinae huius theorematis demonstratio, quae etsi veritatem enunciatam rite ostendat, non tamen ex genuinis principiis hausta videtur. Unde innumerabilia alia eiusdem generis in ellipsi aliisque lineis curvis invenire licet. Idemque ex eo vel maxime apparet, quod Auctor huius demonstrationis solutionem problematis aggredi non sit ausus, cum tamen ex iisdem principiis nostri Auctoris expediri queat. In eo autem quaeritur modus in quadrante elliptico partem geometrice assignandi, quae exacte semissi quadrantis aequetur. Celeberrimus igitur EULERUS in hoc scripto non solum suo more theorema memoratum demonstrat, sed etiam problema hoc

1) Vide p. 56. A. K.

2) CH. BOSSUT, *Démonstration d'un théoreme de géométrie énoncé dans les actes de Leipsick année 1754*, Mém. prés. par div. sav. Paris. T. 3, 1760, p. 314. A. K.

resolvit idque ope methodi illius novae, quam iam pridem[1]) in hunc finem excogitavit et cuius bina nova in hoc volumine specimina edidit, quorum occasione de ista methodo iam fusius est expositum, quae hic repetere superfluum foret. Adiungit etiam alia quaedam non minus notatu digna, veluti id, quod circa finem affert, quo in ellipsi arcus assignatur, qui sit totius perimetri ellipticae pars tertia.

Theorema istud et problema versantur circa arcus ellipticos; illo semissis ellipseos quaeque ita secatur, ut partium differentia sit geometrice assignabilis, hoc vero constructio geometrica arcus postulatur, qui sit semissis quadrantis elliptici. Tam demonstratio theorematis quam solutio problematis sequuntur ex iis, quae iam aliquoties[1]) de comparatione linearum curvarum praelegi; et quoniam methodus, qua hoc argumentum pertractavi, non solum nova, sed etiam plurimum recondita videbatur, has propositiones ideo publicare constitueram, ut alii quoque vires suas in iis evolvendis exercerent novisque methodis, quibus forte eo pertingerent, fines Analyseos amplificarent. Cum autem nemo adhuc sit inventus, qui hoc negotium cum successu susceperit, etiamsi vix dubitare liceat, quin plures id frustra tentaverint, merito mihi quidem inde concludere videor praeter methodum, qua ego sum usus, vix ullam aliam viam ad huiusmodi speculationes patere. Quia enim haec methodus perquam indirecte et quasi per ambages procedit neque verisimile sit eam cuiquam, qui huiusmodi problemata sit aggressurus, unquam in mentem venire, mirum non est has quaestiones ab aliis intactas esse relictas. Etsi igitur iam aliquot specimina huius methodi singularis ediderim, tamen operae pretium fore arbitror, si eius explicationem magis illustravero atque ad enodationem problematis ac theorematis propositi accuratius accommodavero, ut ea saepius tractando magis trita et familiaris reddatur. Cum enim eius ope ad maxime absconditas proprietates ellipsis aliarumque curvarum quasi inopinato sim deductus, nullum est dubium, quin in ea plurima alia profundissimae indaginis contineantur, quae non nisi post frequentiorem tractationem inde eruere liceat.

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1) L. EULERI Commentationes 252, 263, 261 (indicis ENESTROEMIANI); vide p. 80, 108, 153. A. K.

LEMMA 1

1. *Si binae variabiles x et y ita a se invicem pendeant, ut sit*

$$0 = \alpha + \beta(xx + yy) + 2\gamma xy + \delta xxyy,$$

erit sive summa sive differentia harum formularum integralium

$$\int \frac{dy}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}} \pm \int \frac{dx}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}}$$

aequalis quantitati constanti.

DEMONSTRATIO

Cum enim sit

$$0 = \alpha + \beta(xx + yy) + 2\gamma xy + \delta xxyy,$$

erit inde utramque radicem extrahendo

$$y = \frac{-\gamma x \pm \sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}}{\beta + \delta xx},$$

$$x = \frac{-\gamma y \pm \sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}}{\beta + \delta yy},$$

unde sequitur fore

$$\beta y + \gamma x + \delta xxy = \pm\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)},$$

$$\beta x + \gamma y + \delta xyy = \pm\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}.$$

Quodsi vero aequatio proposita differentietur, orietur

$$0 = \beta xdx + \beta ydy + \gamma ydx + \gamma xdy + \delta xyydx + \delta xxydy$$

seu

$$0 = dx(\beta x + \gamma y + \delta xyy) + dy(\beta y + \gamma x + \delta xxy),$$

quae abit in hanc

$$\frac{dy}{\beta x + \gamma y + \delta xyy} + \frac{dx}{\beta y + \gamma x + \delta xxy} = 0.$$

Substituantur loco denominatorum formulae illae irrationales, ut prodeant duo membra differentialia, in quibus variabiles x et y sint a se invicem separatae, ac sumendis integralibus obtinebitur

$$\int \frac{dy}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)yy - \beta\delta y^4)}} \pm \int \frac{dx}{\sqrt{(-\alpha\beta + (\gamma\gamma - \alpha\delta - \beta\beta)xx - \beta\delta x^4)}} = \text{Const.}$$

COROLLARIUM 1

2. Summa harum formularum integralium erit constans, si in utraque radicis extractione signis radicalibus paria tribuantur signa; sin autem signa statuantur disparia, tum differentia formularum integralium erit constans.

COROLLARIUM 2

3. Si ponamus

$$-\alpha\beta = Ak, \quad \gamma\gamma - \alpha\delta - \beta\beta = Bk, \quad -\beta\delta = Ck,$$

inde fiet

$$\alpha = \frac{-Ak}{\beta}, \quad \delta = \frac{-Ck}{\beta} \quad \text{et} \quad \gamma = \frac{\sqrt{(ACkk + Bk\beta\beta + \beta^4)}}{\beta}.$$

Quare si relatio inter x et y hac aequatione exprimatur

$$0 = -Ak + \beta\beta(xx + yy) + 2xy\sqrt{(ACkk + Bk\beta\beta + \beta^4)} - Ckxxyy,$$

erit

$$\int \frac{dy}{\sqrt{(A + Byy + Cy^4)}} \pm \int \frac{dx}{\sqrt{(A + Bxx + Cx^4)}} = \text{Const.}$$

COROLLARIUM 3

4. Substitutis autem loco α, δ, γ his valoribus erit

$$y = \frac{-x\sqrt{(ACkk + Bk\beta\beta + \beta^4)} \pm \beta\sqrt{k(A + Bxx + Cx^4)}}{\beta\beta - Ckxx},$$

$$x = \frac{-y\sqrt{(ACkk + Bk\beta\beta + \beta^4)} \pm \beta\sqrt{k(A + Byy + Cy^4)}}{\beta\beta - Ckyy},$$

qui ergo sunt valores illi aequationi integrali convenientes, et quia in his formulis inest constans arbitraria $\frac{\beta\beta}{k}$, eae integrale completum exhibere sunt censendae.

COROLLARIUM 4

5. Ad has formulas commodiores reddendas, quia posito $x = 0$ fit $y = \pm\frac{\sqrt{Ak}}{\beta}$, ponatur $\frac{\sqrt{Ak}}{\beta} = f$ et prodibit

$$y = \frac{x\sqrt{A(A + Bff + Cf^4)} \pm f\sqrt{A(A + Bxx + Cx^4)}}{A - Cffxx},$$

$$x = \frac{y\sqrt{A(A + Bff + Cf^4)} \pm f\sqrt{A(A + Byy + Cy^4)}}{A - Cffyy},$$

quae sunt radices huius aequationis

$$0 = -Aff + A(xx + yy) - 2xy\sqrt{A(A + Bff + Cf^4)} - Cffxxyy.$$

COROLLARIUM 5

6. Si ergo relatio inter x et y hac aequatione exprimatur

$$0 = -Aff + A(xx + yy) \pm 2xy\sqrt{A(A + Bff + Cf^4)} - Cffxxyy,$$

tum erit

$$\int\frac{dy}{\sqrt{(A + Byy + Cy^4)}} \pm \int\frac{dx}{\sqrt{(A + Bxx + Cx^4)}} = \text{Const.}$$

seu

$$\frac{dy}{\sqrt{(A + Byy + Cy^4)}} \pm \frac{dx}{\sqrt{(A + Bxx + Cx^4)}} = 0.$$

COROLLARIUM 6

7. Vicissim ergo si habeatur haec aequatio differentialis

$$\frac{dy}{\sqrt{(A + Byy + Cy^4)}} + \frac{dx}{\sqrt{(A + Bxx + Cx^4)}} = 0,$$

relatio inter x et y ita se habebit, ut sit

$$y = \frac{-x\sqrt{A(A + Bff + Cf^4)} + f\sqrt{A(A + Bxx + Cx^4)}}{A - Cffxx}$$

seu

$$x = \frac{-y\sqrt{A(A + Bff + Cf^4)} + f\sqrt{A(A + Byy + Cy^4)}}{A - Cffyy}.$$

COROLLARIUM 7

8. Verum proposita hac aequatione differentiali

$$\frac{dy}{\sqrt{(A + Byy + Cy^4)}} - \frac{dx}{\sqrt{(A + Bxx + Cx^4)}} = 0$$

aequatio integralis completa erit

$$y = \frac{x\sqrt{A(A + Bff + Cf^4)} + f\sqrt{A(A + Bxx + Cx^4)}}{A - Cffxx}$$

seu

$$x = \frac{y\sqrt{A(A + Bff + Cf^4)} - f\sqrt{A(A + Byy + Cy^4)}}{A - Cffyy}.$$

SCHOLION

9. Retinebo determinationes huius postremi casus, quibus efficitur, quodsi relatio inter binas variabiles x et y fuerit

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy$$

sive

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

et

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy},$$

tum hanc aequationem differentialem locum habere

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}} - \frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = 0$$

seu sumtis integralibus fore

$$\int\frac{dy}{\sqrt{(A+Byy+Cy^4)}} - \int\frac{dx}{\sqrt{(A+Bxx+Cx^4)}} = \text{Const.}$$

Pro hoc ergo casu erit

$$\sqrt{(A+Bxx+Cx^4)} = \frac{y(A-Cffxx) - x\sqrt{A(A+Bff+Cf^4)}}{f\sqrt{A}}$$

et

$$\sqrt{(A+Byy+Cy^4)} = \frac{-x(A-Cffyy) + y\sqrt{A(A+Bff+Cf^4)}}{f\sqrt{A}}$$

sicque fiet

$$\frac{fdy\sqrt{A}}{y\sqrt{A(A+Bff+Cf^4)} - x(A-Cffyy)} + \frac{fdx\sqrt{A}}{x\sqrt{A(A+Bff+Cf^4)} - y(A-Cffxx)} = 0.$$

LEMMA 2

10. *Eadem manente relatione inter binas variabilis x et y, ut sit*

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy$$

seu

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

et

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy},$$

erit differentia harum formularum integralium

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}}-\int\frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}$$

geometrice assignabilis.

DEMONSTRATIO

Ad hoc ostendendum ponamus hanc differentiam $= V$, ut sit

$$\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}}-\frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}=dV.$$

Quare cum sit

$$\frac{dy}{\sqrt{(A+Byy+Cy^4)}}=\frac{dx}{\sqrt{(A+Bxx+Cx^4)}},$$

erit

$$dV=\frac{\mathfrak{B}(yy-xx)dx}{\sqrt{(A+Bxx+Cx^4)}}=\frac{\mathfrak{B}f(yy-xx)dx\sqrt{A}}{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}}.$$

Ponamus iam $xy=u$, ut sit $y=\frac{u}{x}$ et

$$0=-Aff+Axx+\frac{Auu}{xx}-2u\sqrt{A(A+Bff+Cf^4)}-Cffuu,$$

qua aequatione differentiata fit

$$0=Axdx-\frac{Auudx}{x^3}+\frac{Audu}{xx}-du\sqrt{A(A+Bff+Cf^4)}-Cffudu;$$

unde ob $\frac{u}{x}=y$ per x multiplicando oritur

$$\frac{dx}{y(A-Cffxx)-x\sqrt{A(A+Bff+Cf^4)}}=\frac{du}{A(yy-xx)},$$

quae multiplicata per $\mathfrak{B}f(yy-xx)\sqrt{A}$ praebet

$$dV=\frac{\mathfrak{B}fdu}{\sqrt{A}} \quad \text{et} \quad V=\text{Const.}+\frac{\mathfrak{B}fxy}{\sqrt{A}}.$$

Quam ob rem pro formularum integralium differentia habebimus

$$\int\frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}}-\int\frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}=\text{Const.}+\frac{\mathfrak{B}fxy}{\sqrt{A}},$$

quae utique est geometrice assignabilis.

COROLLARIUM 1

11. Propositis ergo duabus formulis integralibus similibus

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy)}{\sqrt{(A+Byy+Cy^4)}} \quad \text{et} \quad \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx)}{\sqrt{(A+Bxx+Cx^4)}}$$

eiusmodi relatio inter x et y exhiberi potest, ut harum formularum differentia fiat geometrice assignabilis.

COROLLARIUM 2

12. Hunc scilicet in finem talis relatio inter variabiles x et y statui debet, ut sit

$$0 = -Aff + A(xx+yy) - 2xy\sqrt{A(A+Bff+Cf^4)} - Cffxxyy,$$

cuius aequationis resolutio cum sit ambigua, capi debet

$$y = \frac{x\sqrt{A(A+Bff+Cf^4)} + f\sqrt{A(A+Bxx+Cx^4)}}{A-Cffxx}$$

et

$$x = \frac{y\sqrt{A(A+Bff+Cf^4)} - f\sqrt{A(A+Byy+Cy^4)}}{A-Cffyy}.$$

COROLLARIUM 3

13. Quemadmodum hic y per x et f atque x per y et f definitur, ita etiam simili modo f per x et y definiri potest. Erit enim

$$f = \frac{y\sqrt{A(A+Bxx+Cx^4)} - x\sqrt{A(A+Byy+Cy^4)}}{A-Cxxyy},$$

unde patet, si sit $x=0$, fore $y=f$, ex quo casu constans illa in valorem ipsius V ingrediens definiri debet.

SCHOLION

14. Simili modo demonstrari potest etiam harum formularum integralium differentiam

$$\int \frac{dy(\mathfrak{A}+\mathfrak{B}yy+\mathfrak{C}y^4+\mathfrak{D}y^6)}{\sqrt{(A+Byy+Cy^4)}} - \int \frac{dx(\mathfrak{A}+\mathfrak{B}xx+\mathfrak{C}x^4+\mathfrak{D}x^6)}{\sqrt{(A+Bxx+Cx^4)}} = V$$

esse geometrice assignabilem. Posito enim $xy = u$ erit

$$dV = \frac{fdu}{(yy-xx)\sqrt{A}}(\mathfrak{B}(yy-xx) + \mathfrak{C}(y^4 - x^4) + \mathfrak{D}(y^6 - x^6))$$

ideoque

$$dV = \frac{fdu}{\sqrt{A}}(\mathfrak{B} + \mathfrak{C}(yy+xx) + \mathfrak{D}(y^4 + xxyy + x^4)).$$

At ex aequatione canonica habemus

$$xx + yy = \frac{Aff + 2u\sqrt{A(A + Bff + Cf^4)} + Cffuu}{A}.$$

Ponamus brevitatis gratia $\sqrt{A(A + Bff + Cf^4)} = Fff$, ut sit

$$xx + yy = \frac{ff}{A}(A + 2Fu + Cuu),$$

eritque ob $y^4 + xxyy + x^4 = (xx+yy)^2 - uu$

$$dV = \frac{fdu}{\sqrt{A}}\left\{\begin{array}{c} \mathfrak{B} + \dfrac{\mathfrak{C}ff}{A}(A + 2Fu + Cuu) \\ + \dfrac{\mathfrak{D}f^4}{AA}(A + 2Fu + Cuu)^2 - \mathfrak{D}uu \end{array}\right\}$$

ideoque integrando

$$V = \frac{f}{\sqrt{A}}\left\{\begin{array}{c} \mathfrak{B}u + \dfrac{\mathfrak{C}ff}{A}(Au + Fuu + \frac{1}{3}Cu^3) - \frac{1}{3}\mathfrak{D}u^3 \\ + \dfrac{\mathfrak{D}f^4}{AA}(AAu + 2AFuu + \frac{2}{3}(AC + 2FF)u^3 + CFu^4 + \frac{1}{5}CCu^5) \end{array}\right\}$$

Verum pro praesenti instituto, quo ellipsis nobis est proposita, formulae in lemmate exhibitae sufficiunt.

LEMMA 3

15. *Si* C (Fig. 1) *sit centrum ellipseos eiusque semiaxes* $CA = a$, $CB = b$ *atque ad verticem* A *ducatur tangens* AD, *in qua sumatur portio indefinita* $AZ = z$, *et ex* Z *ad* AD *perpendicularis erigatur* ZMV, *erit arcus huic abscissae* $AZ = z$ *respondens*

$$AM = \int \frac{dz}{b}\sqrt{\frac{b^4 - (bb - aa)zz}{bb - zz}}.$$

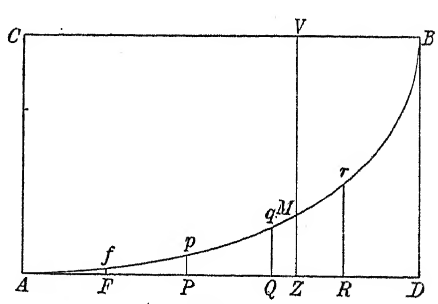

Fig. 1.

DEMONSTRATIO

Ponatur $ZM = v$ et ipse arcus $AM = s$; erit ex natura ellipsis

$$VM = a - v = \frac{a}{b}\sqrt{(bb - zz)}$$

hincque

$$v = a - \frac{a}{b}\sqrt{(bb - zz)} \quad \text{et} \quad dv = \frac{azdz}{b\sqrt{(bb-zz)}}.$$

Quare cum sit $ds = \sqrt{(dz^2 + dv^2)}$, erit

$$ds = dz\sqrt{\left(1 + \frac{aazz}{bb(bb-zz)}\right)} = \frac{dz}{b}\sqrt{\frac{b^4-(bb-aa)zz}{bb-zz}}$$

et integrando

$$s = \text{Arc. } AM = \int\frac{dz}{b}\sqrt{\frac{b^4-(bb-aa)zz}{bb-zz}}$$

integrali ita accepto, ut evanescat posito $z = 0$.

COROLLARIUM 1

16. Ad hanc formulam contrahendam ponamus hic et in sequentibus perpetuo $\frac{bb-aa}{bb} = n$, ut sit $a = b\sqrt{(1-n)}$, eritque arcus abscissae $AZ = z$ respondens

$$AM = \int dz\sqrt{\frac{bb-nzz}{bb-zz}}.$$

Seu cum sit

$$AM = \int\frac{dz(bb-nzz)}{\sqrt{(b^4-(n+1)bbzz+nz^4)}},$$

haec expressio ad nostram formam tractatam

$$\int\frac{dz(\mathfrak{A}+\mathfrak{B}zz)}{\sqrt{(A+Bzz+Cz^4)}}$$

reducetur ponendo

$$\mathfrak{A} = bb, \quad \mathfrak{B} = -n, \quad A = b^4, \quad B = -(n+1)bb, \quad C = n,$$

ita ut sit

$$\sqrt{(A+Bzz+Cz^4)} = \sqrt{(bb-zz)(bb-nzz)}.$$

COROLLARIUM 2

17. Cum ob $a = b\sqrt{(1-n)}$ sit

$$dv = \frac{zdz\sqrt{(1-n)}}{\sqrt{(bb-zz)}} \quad \text{et} \quad ds = dz\sqrt{\frac{bb-nzz}{bb-zz}},$$

erit anguli AMZ sinus $= \frac{dz}{ds} = \sqrt{\frac{bb-zz}{bb-nzz}}$, cosinus $= \frac{dv}{ds} = \frac{z\sqrt{(1-n)}}{\sqrt{(bb-nzz)}}$ et tangens $= \frac{dz}{dv} = \frac{\sqrt{(bb-zz)}}{z\sqrt{(1-n)}}$, quas formulas probe notasse iuvabit:

$$\text{sinus} \quad AMZ = \sqrt{\frac{bb-zz}{bb-nzz}},$$

$$\text{cosinus} \quad AMZ = \frac{z\sqrt{(1-n)}}{\sqrt{(bb-nzz)}},$$

$$\text{tangens} \quad AMZ = \frac{\sqrt{(bb-zz)}}{z\sqrt{(1-n)}}.$$

COROLLARIUM 3

18. Designabo porro arcum AM, qui abscissae cuique $AZ = z$ respondet, hac expressione $\Pi : z$, ut sit

$$AM = \Pi : z = \int dz\sqrt{\frac{bb-nzz}{bb-zz}}.$$

Hinc si variae abscissae ponantur

$$AF = f, \quad AP = p, \quad AQ = q, \quad AR = r, \quad AD = CB = b,$$

erunt arcus respondentes

$$Af = \Pi : f, \quad Ap = \Pi : p, \quad Aq = \Pi : q, \quad Ar = \Pi : r, \quad AMB = \Pi : b.$$

COROLLARIUM 4

19. Hoc modo etiam arcus, qui non in puncto A terminantur, commode exprimi poterunt; sic enim erit

$$\text{arcus } fp = \Pi : p - \Pi : f, \quad \text{arcus } pq = \Pi : q - \Pi : p,$$

$$\text{arcus } qr = \Pi : r - \Pi : q, \quad \text{arcus } pr = \Pi : r - \Pi : p,$$

item

$$\text{arcus } Bp = \Pi : b - \Pi : p, \quad \text{arcus } Bq = \Pi : b - \Pi : q.$$

Denotat enim $\Pi : b$ arcum totius quadrantis AMB ideoque $4\Pi : b$ totam ellipsis peripheriam.

PROBLEMA 1

20. *Proposito in ellipsi arcu Af* (Fig. 1, p. 209) *in vertice A terminato ab alio quovis puncto p arcum abscindere pq, qui ab illo arcu Af discrepet quantitate geometrice assignabili.*

SOLUTIO

Positis abscissis, quae punctis f, p et q respondent, $AF=f$, $AP=p$, et $AQ=q$ ex datis f et p convenienter determinari oportet q. Cum igitur pro lemmate secundo sit

$$\mathfrak{A}=bb, \quad \mathfrak{B}=-n, \quad A=b^4, \quad B=-(n+1)bb \quad \text{et} \quad C=n,$$

capiatur q ita, ut sit

$$q=\frac{bbp\sqrt{(bb-ff)(bb-nff)}+bbf\sqrt{(bb-pp)(bb-npp)}}{b^4-nffpp},$$

eritque per lemmatis conclusionem

$$\int dq\sqrt{\frac{bb-nqq}{bb-qq}}-\int dp\sqrt{\frac{bb-npp}{bb-pp}}=\text{Const.}-\frac{nfpq}{bb}.$$

At est

$$\int dq\sqrt{\frac{bb-nqq}{bb-qq}}=\Pi:q \quad \text{et} \quad \int dp\sqrt{\frac{bb-npp}{bb-pp}}=\Pi:p,$$

unde

$$\Pi:q-\Pi:p=\text{Const.}-\frac{nfpq}{bb},$$

ubi tantum superest, ut constans debite definiatur. Verum quia posito $p=0$ fit $q=f$, ad quem casum aequatione translata fiet $\Pi:f=\text{Const.}$, quo valore introducto habebimus

$$\Pi:q-\Pi:p=\Pi:f-\frac{nfpq}{bb}$$

sive

$$\text{Arc.}\,pq=\text{Arc.}\,Af-\frac{nfpq}{bb}.$$

COROLLARIUM 1

21. Quia vero eidem abscissae $AQ=q$ bina in ellipsi puncta q respondent, ad hoc punctum perfecte determinandum etiam applicatae Qq magnitudo definiri debet. Est vero

$$Qq = a - \frac{a}{b}\sqrt{(bb-qq)} = \left(b - \sqrt{(bb-qq)}\right)\sqrt{(1-n)}$$

et

$$\sqrt{(bb-qq)} = \frac{b^3\sqrt{(bb-ff)(bb-pp)} - bfp\sqrt{(bb-nff)(bb-npp)}}{b^4 - nffpp}.$$

Tum etiam notari meretur

$$\sqrt{(bb-nqq)} = \frac{b^3\sqrt{(bb-nff)(bb-npp)} - nbfp\sqrt{(bb-ff)(bb-pp)}}{b^4 - nffpp};$$

si igitur valor ipsius $\sqrt{(bb-qq)}$ fit negativus, punctum q in superiori ellipsis quadrante capi debet.

COROLLARIUM 2

22. Hic igitur primo relatio notari debet, quae inter tria puncta f, p et q intercedit, quae ita est comparata, ut ex binis datis tertium inveniri possit.

I. Si f et p sint data, erit

$$q = \frac{bbp\sqrt{(bb-ff)(bb-nff)} + bbf\sqrt{(bb-pp)(bb-npp)}}{b^4 - nffpp},$$

$$\sqrt{(bb-qq)} = \frac{b^3\sqrt{(bb-ff)(bb-pp)} - bfp\sqrt{(bb-nff)(bb-npp)}}{b^4 - nffpp},$$

$$\sqrt{(bb-nqq)} = \frac{b^3\sqrt{(bb-nff)(bb-npp)} - nbfp\sqrt{(bb-ff)(bb-pp)}}{b^4 - nffpp}.$$

II. Si f et q sint data, erit

$$p = \frac{bbq\sqrt{(bb-ff)(bb-nff)} - bbf\sqrt{(bb-qq)(bb-nqq)}}{b^4 - nffqq},$$

$$\sqrt{(bb-pp)} = \frac{b^3\sqrt{(bb-ff)(bb-qq)} + bfq\sqrt{(bb-nff)(bb-nqq)}}{b^4 - nffqq},$$

$$\sqrt{(bb-npp)} = \frac{b^3\sqrt{(bb-nff)(bb-nqq)} + nbfq\sqrt{(bb-ff)(bb-qq)}}{b^4 - nffqq}.$$

III. Si p et q sint data, erit

$$f = \frac{bbq\sqrt{(bb-pp)(bb-npp)} - bbp\sqrt{(bb-qq)(bb-nqq)}}{b^4 - nppqq},$$

$$\sqrt{(bb-ff)} = \frac{b^3\sqrt{(bb-pp)(bb-qq)} + bpq\sqrt{(bb-npp)(bb-nqq)}}{b^4 - nppqq},$$

$$\sqrt{(bb-nff)} = \frac{b^3\sqrt{(bb-npp)(bb-nqq)} + nbpq\sqrt{(bb-pp)(bb-qq)}}{b^4 - nppqq}.$$

Hae autem formulae omnes ex hac nascuntur

$$0 = -b^4ff + b^4pp + b^4qq - 2bbpq\sqrt{(bb-ff)(bb-nff)} - nffppqq,$$

quae adeo ad hanc rationalem, in qua f, p et q aequaliter insunt, reducitur

$$\begin{aligned}0 = b^8(f^4 + p^4 + q^4) + 4(n+1)b^6ffppqq - 2b^8(ffpp + ffqq + ppqq)\\ - 2nb^4ffppqq(ff + pp + qq) + nnf^4p^4q^4.\end{aligned}$$

COROLLARIUM 3

23. Harum formularum igitur ope, si trium punctorum f, p et q data sint bina quaecunque, tertium inveniri poterit, ut arcuum Af et pq differentia geometrice fiat assignabilis. Erit enim

$$\text{Arc. } Af - \text{Arc. } pq = \text{Arc. } Ap - \text{Arc. } fq = \frac{nfpq}{bb}.$$

COROLLARIUM 4

24. Denotat autem b semiaxem ellipsis CB et posito altero $CA = a$ fecimus $\frac{bb-aa}{bb} = n$; unde, si $n = 0$, ellipsis abit in circulum et arcuum assignatorum differentia evanescit. Ellipsis autem abibit in parabolam, cuius semiparameter $= c$, si $bb = ac$ et $a = \infty$. Hoc ergo casu fiet

$$n = \frac{c-a}{c} = -\frac{a}{c} \quad \text{et} \quad \frac{n}{bb} = -\frac{1}{cc}$$

ideoque

$$n = -\frac{bb}{cc} \quad \text{et} \quad \sqrt{(bb-ff)} = b, \quad \sqrt{(bb-nff)} = b\sqrt{\left(1-\frac{ff}{cc}\right)};$$

unde formulae superiores ad parabolam transferri poterunt.

COROLLARIUM 5

25. Si easdem formulas ad hyperbolam accommodare velimus, semiaxem b ita imaginarium statui oportet, ut eius quadratum bb fiat quantitas negativa. Seu, quod eodem redit, in nostris formulis ubique loco bb scribatur $-bb$ et semiaxis a capiatur negative; tum vero n erit numerus unitate maior.

PROBLEMA 2

26. *In quadrante elliptico AB* (Fig. 2) *dato puncto quocunque f invenire aliud punctum g, ut arcuum Af et Bg differentia sit geometrice assignabilis.*

SOLUTIO

Ex praecedente problemate hoc facile resolvitur; positis enim semiaxibus $CA = a$, $CB = b$ et $\frac{bb - aa}{bb} = n$ punctum q in praecedente problemate in B usque promoveri oportet, ut fiat $q = b$; tum sint abscissae super tangente AD vel axe CB sumtae punctis f et g respondentes $AF = C\mathfrak{F} = f$ et $AG = C\mathfrak{G} = g$, ita ut, quod ante erat p, nunc sit g, atque ex dato puncto f determinatio puncti g per formulas § 22 ita se habebit ob $p = g$ et $q = b$

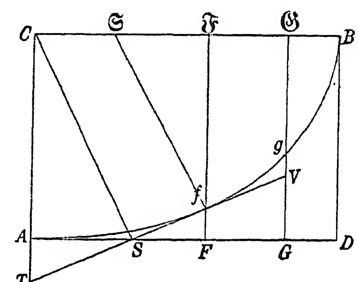

Fig. 2.

$$g = \frac{b^3\sqrt{(bb-ff)(bb-nff)}}{b^4 - nbbff} = b\sqrt{\frac{bb-ff}{bb-nff}},$$

$$\sqrt{(bb-gg)} = \frac{bbf\sqrt{(bb-nff)(bb-nbb)}}{b^4 - nbbff} = \frac{bf\sqrt{(1-n)}}{\sqrt{(bb-nff)}},$$

$$\sqrt{(bb-ngg)} = \frac{b^3\sqrt{(bb-nff)(bb-nbb)}}{b^4 - nbbff} = \frac{bb\sqrt{(1-n)}}{\sqrt{(bb-nff)}}.$$

Unde si anguli, quos applicatae Ff et Gg cum curva faciunt, in computum ducantur, erit

$$g = b \sin. AfF \quad \text{et} \quad f = b \sin. AgG.$$

Atque hinc sequitur ista constructio pro puncto g inveniendo: Ad punctum f ducatur tangens fT, donec axi CA producto occurrat in T, tum in ea, si opus est, producta capiatur $TV = CB = b$ et per V agatur recta $\mathfrak{G}G$ axi CA parallela eritque punctum g quaesitum, ita ut arcuum Af et Bg differentia sit geometrice assignabilis. Verum ex problemate praecedente ob $p = g$ et $q = b$ erit haec differentia

$$\text{Arc. } Af - \text{Arc. } Bg = \frac{nfg}{b} = nf\sqrt{\frac{bb-ff}{bb-nff}}.$$

Ad quam construendam notetur esse

$$Tf = \frac{AF}{\sin. AfF} = f\sqrt{\frac{bb-nff}{bb-ff}}$$

et ex natura ellipsis

$$CT = \frac{ab}{\sqrt{(bb-ff)}} = \frac{bb\sqrt{(1-n)}}{\sqrt{(bb-ff)}}.$$

Hinc si ex centro ellipsis C in tangentem Tf demittatur perpendiculum CS, ob ang. $CTS =$ ang. AfF eiusque sinum $= \sqrt{\frac{bb-ff}{bb-nff}}$ et cosinum $= \frac{f\sqrt{(1-n)}}{\sqrt{(bb-nff)}}$ erit

$$TS = CT \cos. CTS = \frac{bbf(1-n)}{\sqrt{(bb-ff)(bb-nff)}}$$

hincque

$$Sf = Tf - TS = \frac{bbf - nf^3 - bbf + nbbf}{\sqrt{(bb-ff)(bb-nff)}} = \frac{nf(bb-ff)}{\sqrt{(bb-ff)(bb-nff)}} = nf\sqrt{\frac{bb-ff}{bb-nff}}.$$

Portio igitur tangentis fS inter perpendiculum CS et punctum contactus f contenta praebebit differentiam arcuum Af et Bg, ita ut sit

$$\text{Arc. } Af - \text{Arc. } Bg = \text{Arc. } Ag - \text{Arc. } Bf = Sf.$$

COROLLARIUM 1

27. Haec differentia arcuum facilius inveniri potest, si in f ad tangentem ducatur normalis $f\mathfrak{S}$; tum enim ex natura ellipsis statim constat esse

$$C\mathfrak{S} = f - \frac{aa}{bb}f = nf.$$

Quare cum CS ipsi $\mathfrak{S}f$ sit parallela et angulus $BCS = CTS = TfF$ eiusque ergo sinus $= \sqrt{\frac{bb-ff}{bb-nff}}$, erit

$$Sf = C\mathfrak{S} \sin. BCS = nf\sqrt{\frac{bb-ff}{bb-nff}}.$$

COROLLARIUM 2

28. Simili modo ex puncto g definietur punctum f; si enim ad g ducatur tangens usque ad axem CA atque ab intersectione eius cum axe in ea capiatur portio alteri semiaxi CB aequalis, haec praecise in recta Ff terminabitur ideoque punctum f monstrabit.

COROLLARIUM 3

29. Constructio ergo puncti g ex dato puncto f ita se habebit: Ad punctum f ducatur tangens axi CA producto occurrens in T in eaque a T abscindatur portio TV semiaxi CB aequalis, et recta $G\mathfrak{G}$ axi CA parallela per punctum V acta in ellipsi punctum quaesitum g definiet. Tum enim, si ex centro ellipsis C in illam tangentem perpendiculum CS demittatur, erit

$$\text{Arc. } Af - \text{Arc. } Bg = \text{Rectae } Sf$$

hincque etiam

$$\text{Arc. } Af - \text{Recta } fS = \text{Arc. } Bg.$$

COROLLARIUM 4

30. Casus notabilis est, quo bina puncta f et g in unum colliquescunt, ita ut arcus quadrantis AfB (Fig. 3) in puncto f ita secari iubeatur, ut partium Af et Bf differentia fiat geometrice assignabilis. Hunc in finem ponatur in solutione $g = f$, unde fit

$$f = b\sqrt{\frac{bb - ff}{bb - nff}}$$

hincque

$$2bbff - nf^4 = b^4 \quad \text{et} \quad \frac{bb}{ff} = 1 + \sqrt{(1-n)} = \frac{a+b}{b}.$$

Quare pro puncto hoc f capi debet abscissa

$$AF = f = b\sqrt{\frac{b}{a+b}};$$

atque ob

$$\sqrt{\frac{bb - ff}{bb - nff}} = \frac{f}{b}$$

erit partium differentia

$$Af - Bf = \frac{nff}{b} = \frac{nbb}{a+b},$$

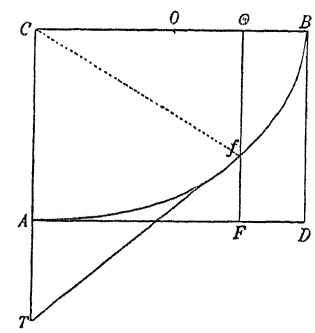

Fig. 3.

quae, cum sit $n = \frac{bb - aa}{bb}$, abit in $Af - Bf = b - a$, ita ut aequalis evadat differentiae semiaxium. Unde puncto f hoc modo definito, ut sit $f = b\sqrt{\frac{b}{a+b}}$, erit etiam

$$AC + Af = BC + Bf$$

seu ducto radio Cf ambo trilinea ACf et BCf pari perimetro includuntur.

COROLLARIUM 5

31. Quia supra habuimus

$$CT = \frac{ab}{\sqrt{(bb-ff)}},$$

erit pro praesenti casu

$$CT = \sqrt{(aa+ab)} \quad \text{ob} \quad ff = \frac{b^3}{a+b};$$

unde sequens concinna puncti f constructio deducitur: Bisecto semiaxe BC in O intervallo $OT = OC + AC$ definiatur in CA producta punctum T, unde intervallo $Tf = BC$ punctum f in ellipsi designetur eritque f punctum quaesitum et recta Tf eius tangens.

PROBLEMA 3

32. *Proposita semiellipsi $AB\alpha$* (Fig. 4) *in eaque sumto quocunque puncto p definire punctum q ita, ut arcus pBq differat a quadrante elliptico ApB quantitate geometrice assignabili.*

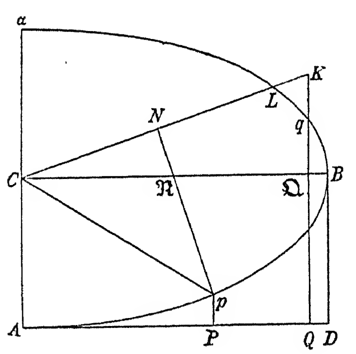

Fig. 4.

SOLUTIO

Positis ut hactenus semiaxibus $CA = a$, $CB = b$ et ad abbreviandum $n = \frac{bb-aa}{bb}$ in solutione problematis primi promoveatur punctum f in B usque eritque vi eius arcuum AB et pq differentia geometrice assignabilis, uti requiritur. Demissis ergo ad tangentem AD ex p et q perpendiculis pP et qQ sint $AP = p$ et $AQ = q$ atque ob $f = b$ habebimus ex § 22

$$q = \frac{b\sqrt{(bb-pp)(bb-npp)}}{bb-npp} = b\sqrt{\frac{bb-pp}{bb-npp}},$$

$$\sqrt{(bb-qq)} = \frac{-p\sqrt{(bb-nbb)(bb-npp)}}{bb-npp} = \frac{-bp\sqrt{(1-n)}}{\sqrt{(bb-npp)}},$$

ʰitatis signum — indicat ulteriorem intersectionem perpendiculi QK · accipi oportere, secus atque in problemate praecedente. Cum $\frac{p}{..}$ exprimat sinum anguli, quem applicata Pp cum curva facit,

erit $q = b$ sin. ApP. Ad Qq, si opus est, productam ex centro C dirigatur recta CK semiaxi $CB = b$ aequalis, ut sit $CK = b$, eritque $\frac{q}{b} = \frac{AQ}{CK} =$ sin. ApP hincque sin. $CKQ =$ sin. ApP et $CKQ = ApP$. Ex quo patet rectam CK parallelam fore tangenti in puncto p. Quare iuncta Cp eaque ut semidiametro spectata erit CL eius semidiameter coniugata, in qua proinde producta, si capiatur $CK = CB$, perpendiculum KQ ad CB demissum in ellipsi definiet punctum q. Quo invento ob

$$f = b \quad \text{et} \quad q = b\sqrt{\frac{bb - pp}{bb - npp}}$$

erit arcuum differentia

$$\text{Arc. } AB - \text{Arc. } pq = \frac{nfpq}{bb} = np\sqrt{\frac{bb - pp}{bb - npp}} = np \text{ sin. } ApP.$$

Ducatur ad ellipsin in p normalis $p\mathfrak{N}$; erit $C\mathfrak{N} = np$ et producta $p\mathfrak{N}$ in N ang. $C\mathfrak{N}N =$ ang. ApP; quare cum haec pN futura sit normalis in diametrum coniugatam CL, erit $CN = np$ sin. ApP; unde demisso ex p in CL perpendiculo intervallum CN aequabitur differentiae illorum arcuum, ita ut sit

$$\text{Arc. } AB - \text{Arc. } pq = CN.$$

COROLLARIUM 1

33. Cum igitur punctum p pro lubitu assumi possit, infiniti arcus pq exhiberi possunt, qui a quadrante AB differunt quantitate geometrice assignabili. Quare etiam hi arcus inter se different quantitate geometrice assignabili.

COROLLARIUM 2

34. Ex dato ergo puncto p punctum q ita definitur: Ad ductam Cp iungatur semidiameter coniugata CL in K producenda, ut fiat CK aequalis semiaxi CB, ad quem ex K perpendiculum demittatur KQ ellipsin secans in q; erit q punctum quaesitum. Atque demisso ex p in CL perpendiculo pN erit $AB - pq = CN$.

COROLLARIUM 3

35. Quoties perpendiculum pN (Fig. 5, p. 220) intra C et K cadit, arcus pq erit minor quadrante AB, contra autem, si ad alteram partem cadit,

maior. Ita si prius punctum in π detur et rectae $C\pi$ conveniat semidiameter coniugata CL, qua producta in K, ut sit $CK = CB$, et ex K ad CB demisso perpendiculo KQ secante ellipsin in q, quia hic perpendiculum $\pi\nu$ in CL demissum ad alteram partem cadit, erit arcus πq — arcu $AB = C\nu$.

THEOREMA DEMONSTRANDUM

36. *Si ellipsis $AB\alpha\beta$* (Fig. 5) *diametro quacunque $p\pi$ fuerit bisecta ad eamque ducatur diameter coniugata $L\lambda$, cuius semissis CL producatur in K, ut fiat CK alteri semiaxi principali CB aequalis, ad quem ex K demittatur perpendiculum KQ ellipsin secans in q, tum ellipsis semiperimeter $pBL\alpha\pi$ ita secabitur in q, ut partium $\pi\alpha q$ et pBq differentia sit geometrice assignabilis. Ductis enim ex p et π ad diametrum coniugatam $L\lambda$ normalibus pN et $\pi\nu$ intervallum $N\nu$ illi differentiae ita aequabitur, ut sit*

$$\text{Arc.}\,\pi\alpha q - \text{Arc.}\,pBq = N\nu.$$

DEMONSTRATIO

Quia CL est semidiameter coniugata conveniens semidiametro Cp, ex constructione, qua punctum q est definitum, patet per § 34 fore

$$\text{Arc.}\,AB - \text{Arc.}\,pq = CN.$$

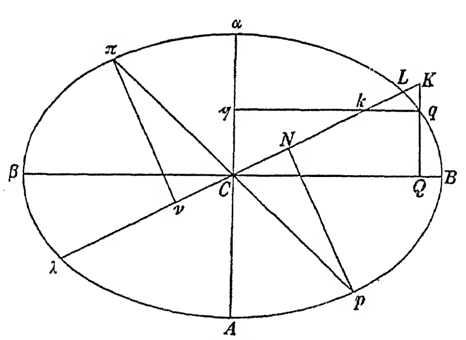

Fig. 5.

Deinde, quia CL est quoque semidiameter coniugata conveniens semidiametro $C\pi$, ex § 35 patet esse

$$\text{Arc.}\,\pi q - \text{Arc.}\,AB = C\nu.$$

Addantur hae duae aequationes ac resultabit

$$\text{Arc.}\,\pi q - \text{Arc.}\,pq = CN + C\nu = N\nu.$$

COROLLARIUM

37. Perinde est, utri semiaxi principali semidiameter CL producta eiusve portio aequalis capiatur, dummodo ex eius termino ad eum ipsum

axem perpendiculum demittatur. Ita in CL potuisset abscindi portio Ck semiaxi minori $C\alpha$ aequalis; recta enim qkq per k ad $C\alpha$ normaliter ducta in ellipsi idem punctum q prodidisset.

SCHOLION

38. En ergo demonstrationem completam theorematis in Actis Erud. Lips. propositi, quae ita est comparata, ut nullo modo ex vulgaribus ellipsis proprietatibus derivari potuisset, neque etiam Analysis infinitorum multum auxilii attulerit, nisi hoc ipso modo, quo hic sum usus, in subsidium vocetur. Ex profundis quidem speculationibus Ill. Comitis FAGNANI hanc quoque demonstrationem deducere liceret; verum inde vix via pateret ad problema ibidem propositum resolvendum, in cuius ergo gratiam sequentia sunt praemittenda.

PROBLEMA 4

39. *Arcum ellipticum quemcunque Ag (Fig. 6) ad alterum axem principalem in A terminatum ita secare in f, ut partium Af et fg differentia sit geometrice assignabilis.*

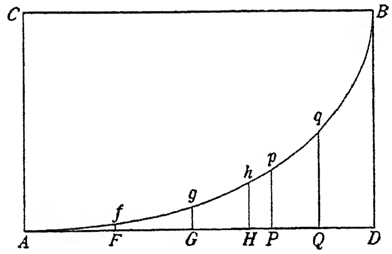

Fig. 6.

SOLUTIO

Positis semiaxibus $CA = a$, $CB = b$ et brevitatis gratia $n = \frac{bb - aa}{bb}$ in verticis A tangente AD sumantur abscissae ac ponatur abscissa toti arcui Ag dato respondens $AG = g$, quaesita autem, quae puncto f respondeat, sit $AF = f$. Cum igitur differentia arcuum Af et fg debeat esse geometrice assignabilis, quaestio continetur in probl. 1 sumendo ibi $p = f$ et ponendo $q = g$, unde obtinebimus has formulas

$$g = \frac{2bbf\sqrt{(bb - ff)(bb - nff)}}{b^4 - nf^4},$$

$$\sqrt{(bb - gg)} = \frac{b^3(bb - ff) - bff(bb - nff)}{b^4 - nf^4} = \frac{b(b^4 - 2bbff + nf^4)}{b^4 - nf^4},$$

$$\sqrt{(bb - ngg)} = \frac{b^3(bb - nff) - nbff(bb - ff)}{b^4 - nf^4} = \frac{b(b^4 - 2nbbff + nf^4)}{b^4 - nf^4}.$$

Ex quibus combinatione oritur

$$\sqrt{(bb-ngg)} - n\sqrt{(bb-gg)} = \frac{(1-n)b(b^4+nf^4)}{b^4-nf^4}$$

hincque

$$\frac{nf^4}{b^4} = \frac{\sqrt{(bb-ngg)}-n\sqrt{(bb-gg)}-(1-n)b}{\sqrt{(bb-ngg)}-n\sqrt{(bb-gg)}+(1-n)b},$$

quae formula reducitur ad

$$\frac{nnf^4}{b^4} = \frac{\left(\sqrt{(bb-ngg)}-n\sqrt{(bb-gg)}-(1-n)b\right)^2}{2bb-(1+n)gg-2\sqrt{(bb-gg)(bb-ngg)}},$$

unde radice quadrata extracta fit

$$\frac{nff}{bb} = \frac{\sqrt{(bb-ngg)}-n\sqrt{(bb-gg)}-(1-n)b}{\sqrt{(bb-ngg)}-\sqrt{(bb-gg)}} = \frac{\left(b-\sqrt{(bb-gg)}\right)\left(b-\sqrt{(bb-ngg)}\right)}{gg},$$

ex qua porro elicimus

$$\frac{bb-nff}{bb} = \frac{(1-n)\left(b-\sqrt{(bb-gg)}\right)}{\sqrt{(bb-ngg)}-\sqrt{(bb-gg)}} = \frac{\left(b-\sqrt{(bb-gg)}\right)\left(\sqrt{(bb-ngg)}+\sqrt{(bb-gg)}\right)}{gg},$$

$$\frac{n(bb-ff)}{bb} = \frac{(1-n)\left(b-\sqrt{(bb-ngg)}\right)}{\sqrt{(bb-ngg)}-\sqrt{(bb-gg)}} = \frac{\left(b-\sqrt{(bb-ngg)}\right)\left(\sqrt{(bb-ngg)}+\sqrt{(bb-gg)}\right)}{gg}.$$

Punctum igitur quaesitum f ita determinabitur, ut sit

$$f = \frac{b}{g\sqrt{n}}\sqrt{\left(b-\sqrt{(bb-gg)}\right)\left(b-\sqrt{(bb-ngg)}\right)},$$

$$\sqrt{(bb-ff)} = \frac{b}{g\sqrt{n}}\sqrt{\left(b-\sqrt{(bb-ngg)}\right)\left(\sqrt{(bb-gg)}+\sqrt{(bb-ngg)}\right)},$$

$$\sqrt{(bb-nff)} = \frac{b}{g}\sqrt{\left(b-\sqrt{(bb-gg)}\right)\left(\sqrt{(bb-gg)}+\sqrt{(bb-ngg)}\right)}.$$

Verum hoc puncto f ita determinato ob $p=f$ et $q=g$ partium inventarum differentia erit

$$\text{Arc. } Af - \text{Arc. } fg = \frac{nffg}{bb} = \frac{\left(b-\sqrt{(bb-gg)}\right)\left(b-\sqrt{(bb-ngg)}\right)}{g}.$$

COROLLARIUM 1

40. Casum huius problematis iam solvimus § 30, quo arcus secandus Ag toti quadranti AB assumitur aequalis. Si enim ponamus $g = b$, reperietur ut ibi

$$f = b\sqrt{\frac{1-\sqrt{(1-n)}}{n}} = b\sqrt{\frac{b(b-a)}{bb-aa}} = \frac{b\sqrt{b}}{\sqrt{(a+b)}}$$

et partium differentia prodit $= b - b\sqrt{(1-n)} = b - a$.

COROLLARIUM 2

41. Si arcus dati Ag alter terminus in superiori quadrante existat eique eadem abscissa $AG = g$ respondeat, eaedem hae formulae valent, nisi quod valor radicalis $\sqrt{(bb-gg)}$ negative capi debeat radicali $\sqrt{(bb-ngg)}$ non mutato.

COROLLARIUM 3

42. Ita si proponatur tota semiperipheria, erit $g = 0$ et $\sqrt{(bb-gg)} = -b$, unde pro hoc casu obtinebitur

$$f = \frac{b}{g\sqrt{n}}\sqrt{2b\left(b - \sqrt{(bb-ngg)}\right)} = b,$$

scilicet arcus Af abibit in quadrantem ellipsis. Sin autem integra ellipsis peripheria proponeretur, tum esset et $g = 0$ et $\sqrt{(bb-gg)} = +b$ sicque valor ipsius f prodiret evanescens, at pro $\sqrt{(bb-ff)}$ capi deberet $-b$.

PROBLEMA 5

43. *Proposito in ellipsi arcu Ag altero termino A in axe principali terminato assignare arcum pq, qui sit praecise semissis arcus dati Ag.*

SOLUTIO

Manentibus superioribus denominationibus sint abscissae punctis p et q respondentes $AP = p$ et $AQ = q$ atque ex puncto p, quasi esset datum, quaeratur q, ut differentia arcuum Af et pq fiat geometrice assignabilis; tum enim quoque differentia arcuum fg et pq geometrice assignari poterit; si-

quidem secundum problema praecedens arcus datus Ag, pro quo est $AG = g$, ita sectus est in f, ut partium Af et fg differentia sit geometrice assignabilis. Hunc ergo in finem esse debet

$$q = \frac{bbp\sqrt{(bb-ff)(bb-nff)} + bbf\sqrt{(bb-pp)(bb-npp)}}{b^4 - nffpp}$$

seu

$$0 = b^4(pp + qq - ff) - 2bbpq\sqrt{(bb-ff)(bb-nff)} - nffppqq.$$

Quo facto erit

$$\text{Arc. } Af - \text{Arc. } pq = \frac{nfpq}{bb}$$

ideoque

$$2\,\text{Arc. } Af - 2\,\text{Arc. } pq = \frac{2nfpq}{bb}.$$

At ex problemate praecedente habemus

$$\text{Arc. } Af - \text{Arc. } fg = \frac{nffg}{bb},$$

qua aequatione ab illa subtracta relinquitur

$$\text{Arc. } Ag - 2\,\text{Arc. } pq = \frac{2nfpq}{bb} - \frac{nffg}{bb}.$$

Quae differentia cum in nihilum abire debeat, habebimus

$$2nfpq = nffg \quad \text{et} \quad 2pq = fg.$$

Pro pq substituatur iste valor $\frac{1}{2}fg$ et obtinebimus

$$b^4(pp + qq) = b^4ff + bbfg\sqrt{(bb-ff)(bb-nff)} + \tfrac{1}{4}nf^4gg$$

existente

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4 - nf^4},$$

vel potius pro f introducatur valor ante inventus

$$f = \frac{b}{g\sqrt{n}}\sqrt{\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-ngg)}\right)},$$

unde fit

$$\sqrt{(bb-ff)(bb-nff)} = \frac{bb\left(\sqrt{(bb-gg)} + \sqrt{(bb-ngg)}\right)}{gg\sqrt{n}}\sqrt{\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-ngg)}\right)}.$$

Postea vero ambae abscissae p et q ex hac aequatione duplicata definiri poterunt

$$pp \pm 2pq + qq = \frac{b^4ff \pm b^4fg + bbfg\sqrt{(bb-ff)(bb-nff)} + \frac{1}{4}nf^4gg}{b^4}$$

vel sublata ista irrationalitate ob

$$bbfg\sqrt{(bb-ff)(bb-nff)} = \frac{1}{2}gg(b^4 - nf^4)$$

habebimus

$$p + q = \frac{\sqrt{\left(b^4ff + b^4fg + \frac{1}{2}b^4gg - \frac{1}{4}nf^4gg\right)}}{bb},$$

$$q - p = \frac{\sqrt{\left(b^4ff - b^4fg + \frac{1}{2}b^4gg - \frac{1}{4}nf^4gg\right)}}{bb},$$

unde utraque abscissa p et q seorsim facile assignatur.

COROLLARIUM 1

44. Si quantitatem subsidiariam f penitus eliminemus, perveniemus ad has duas formulas

$$pp + qq = \frac{1}{4ngg}\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-ngg)}\right)$$
$$\times\left(5bb + 3b\sqrt{(bb-gg)} + 3b\sqrt{(bb-ngg)} + \sqrt{(bb-gg)(bb-ngg)}\right),$$
$$2pq = \frac{b}{\sqrt{n}}\sqrt{\left(b - \sqrt{(bb-gg)}\right)\left(b - \sqrt{(bb-ngg)}\right)}.$$

COROLLARIUM 2

45. Si arcus propositus Ag sit semiperipheriae aequalis ideoque

$$g = 0 \quad \text{et} \quad \sqrt{(bb-gg)} = -b \quad \text{et} \quad \sqrt{(bb-ngg)} = b - \frac{ngg}{2b},$$

fiet pro hoc casu

$$pp + qq = bb \quad \text{et} \quad 2pq = bg = 0$$

ideoque $p = 0$ et $q = b$. Arcus scilicet pq abibit in quadrantem AB, ut natura rei postulat.

PROBLEMA SOLVENDUM

46. *In quadrante elliptico AB* (Fig. 7) *arcum assignare pq, qui praecise sit semissis arcus quadrantis AB.*

SOLUTIO

Ponantur ellipsis semiaxes $CA = a$, $CB = b$ sitque brevitatis gratia $\frac{bb - aa}{bb} = n$. Tum ad A ducatur tangens in eamque ex punctis quaesitis p et q demissa concipiantur perpendicula pP et qQ vocenturque $AP = p$ et $AQ = q$. Iam manifestum est hoc problema esse casum praecedentis, quo punctum g in B assumitur, ita ut hoc sit $g = b$. Quo valore inducto formulae § 44 praebebunt

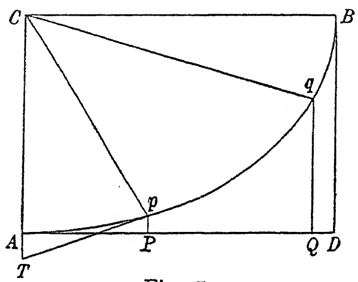

Fig. 7.

$$pp + qq = \frac{1 - \sqrt{(1-n)}}{4n}\left(5bb + 3bb\sqrt{(1-n)}\right)$$

et

$$2pq = bb\sqrt{\frac{1 - \sqrt{(1-n)}}{n}}.$$

At ob

$$n = \frac{bb - aa}{bb} \quad \text{est} \quad \sqrt{(1-n)} = \frac{a}{b} \quad \text{et} \quad \frac{1 - \sqrt{(1-n)}}{n} = \frac{b}{b+a},$$

unde fiet

$$pp + qq = \frac{bb(5b + 3a)}{4(a+b)} \quad \text{et} \quad 2pq = \frac{bb\sqrt{b}}{\sqrt{(a+b)}}$$

hincque

$$q + p = \frac{1}{2}b\sqrt{\frac{5b + 3a + 4\sqrt{b(a+b)}}{a+b}},$$

$$q - p = \frac{1}{2}b\sqrt{\frac{5b + 3a - 4\sqrt{b(a+b)}}{a+b}}$$

ideoque ipsae abscissae erunt

$$AP = \frac{1}{4}b\sqrt{\frac{5b + 3a + 4\sqrt{b(a+b)}}{a+b}} - \frac{1}{4}b\sqrt{\frac{5b + 3a - 4\sqrt{b(a+b)}}{a+b}},$$

$$AQ = \frac{1}{4}b\sqrt{\frac{5b + 3a + 4\sqrt{b(a+b)}}{a+b}} + \frac{1}{4}b\sqrt{\frac{5b + 3a - 4\sqrt{b(a+b)}}{a+b}},$$

qui ambo valores geometrice per circinum et regulam construi possunt.

Haecque est solutio adaequata problematis in Actis Erud. Lipsiensibus propositi.

COROLLARIUM 1

47. Si distantiae binorum punctorum p et q a centro ellipsis desiderentur, notetur posita $AP = p$ fore $Cp = \sqrt{(aa + npp)}$ atque hinc colligitur fore

$$Cp = \frac{\sqrt{(5aa - 2ab + 5bb + (a-b)\sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2}},$$

$$Cq = \frac{\sqrt{(5aa - 2ab + 5bb + (b-a)\sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2}}.$$

COROLLARIUM 2

48. Ambae abscissae p et q etiam hoc modo ad constructionem fortasse aptius exprimi possunt, ut sit

$$AP = p = \frac{b\sqrt{(5b + 3a - \sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2(a+b)}},$$

$$AQ = q = \frac{b\sqrt{(5b + 3a + \sqrt{(9aa + 14ab + 9bb)})}}{2\sqrt{2(a+b)}}.$$

COROLLARIUM 3

49. Si ad puncta p et q tangentes ducantur ad occursum axis CA, magnitudo harum tangentium commode exprimitur. Reperietur enim

$$Tp = \frac{\sqrt{(9aa + 14ab + 9bb)} - 3a - b}{4},$$

pro puncto autem q erit eadem tangens

$$= \frac{\sqrt{(9aa + 14ab + 9bb)} + 3a + b}{4}.$$

COROLLARIUM 4

50. Concipiatur tangens Tp (Fig. 8, p. 228) ad alterum usque axem CB continuata et concursus littera Θ notari eritque permutatis literis a et b

$$\Theta p = \frac{\sqrt{(9aa + 14ab + 9bb)} + a + 3b}{4}$$

ideoque $\Theta p - Tp = a + b$.

COROLLARIUM 5

51. Solutio igitur huius problematis ad hanc quaestionem mere geometricam reducitur:

In quadrante elliptico AB (Fig. 8) *duo eiusmodi puncta* p *et* q *assignare, ita ut ad ea ductis tangentibus* $Tp\Theta$, $tq\theta$, *quoad axibus productis occurrant, sit pro utroque*

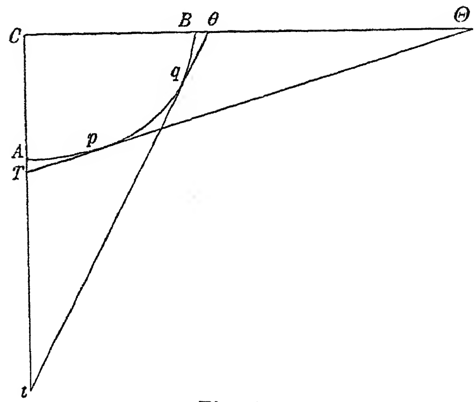

Fig. 8.

$$\Theta p - Tp = CA + CB$$

et

$$tq - \theta q = CA + CB,$$

seu ut differentia partium utriusque tangentis aequalis sit semisummae axium principalium.

Hoc problemate constructo puncta p et q simul ita sunt comparata, ut arcus interceptus pq ad totum quadrantem AB rationem teneat subduplam.

SCHOLION

52. Demonstrato nunc theoremate solutoque problemate, quae in Actis Erud. Lips. extant proposita, antequam huic investigationi finem imponam, problema adhuc multo difficilius pertractabo, quo in ellipsi arcus assignari iubetur, qui totius perimetri ellipseos sit triens. Quoniam enim facillime arcus assignatur, qui totius perimetri sit semissis vel quadrans vel ope problematis praecedentis etiam octans, haud parum notatu dignus videtur casus, quo triens postulatur, cuius solutio, etiamsi ob summam facilitatem, qua res de semissi et quadrante expeditur, non admodum difficilis videatur, tamen ad investigationes perquam prolixas et operosas deducitur, quas superare tentabo.

PROBLEMA 7

53. *Datum ellipsis arcum* Ah (Fig. 6, p. 221) *ad alterum axem principalem in* A *terminatum ita secare in duobus punctis* f *et* g, *ut trium partium* Af, fg *et* gh *binae quaevis quantitate geometrice assignabili discrepent.*

SOLUTIO

Ex punctis f, g, h ad rectam AD, quae ellipsin in A tangit, demissis perpendiculis vocentur abscissae $AF = f$, $AG = g$ et $AH = h$, quarum haec $AH = h$ datur, illas vero duas f et g determinari oportet. Cum autem arcuum Af et fg differentia geometrica esse debeat, erit ex praecedentibus

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4 - nf^4}$$

et

$$Af - fg = \frac{nffg}{bb}.$$

Deinde quia arcuum Af et gh differentia debet esse geometrica, erit per formulas superiores

$$g = \frac{bbh\sqrt{(bb-ff)(bb-nff)} - bbf\sqrt{(bb-hh)(bb-nhh)}}{b^4 - nffhh}$$

et

$$Af - gh = \frac{nfgh}{bb}.$$

Tum igitur quoque tertia differentia erit

$$fg - gh = \frac{nfg}{bb}(h - f).$$

Quodsi iam ambo hi valores ipsius g inter se aequentur, obtinebitur aequatio inter f et h, per quam propterea abscissa f determinabitur, qua inventa porro abscissa g innotescit.

COROLLARIUM 1

54. Aequatis autem duobus valoribus ipsius g eruetur

$$(b^4h - nf^4h - 2b^4f + 2nf^3hh)\sqrt{(bb-ff)(bb-nff)}$$
$$= (b^4f - nf^5)\sqrt{(bb-hh)(bb-nhh)},$$

quae sumtis utrinque quadratis ad duodecimum gradum ascendit.

COROLLARIUM 2

55. Si sit $h = b$ seu arcus Ah in B terminetur, habebitur ista aequatio resolvenda

$$b^5 - nbf^4 - 2b^4f + 2nbbf^3 = 0 \quad \text{seu} \quad nf^4 - 2nbf^3 + 2b^3f - b^4 = 0.$$

PROBLEMA 8

56. *In ellipsi arcum pq* (Fig. 9) *assignare, qui sit tertia pars totius perimetri ellipsis.*

SOLUTIO

Positis semiaxibus $CA = a$, $CB = b$ et brevitatis ergo $n = \frac{bb - aa}{bb}$ dividatur primo tota peripheria ellipsis ita in punctis f et g, ut partium ABf, $f\alpha g$, $g\beta A$ differentiae sint geometrice assignabiles. Statuantur his punctis f et g abscissae respondentes $AF = f$ et $AG = -g$, quatenus haec in plagam oppositam cadit. Problema igitur praecedens ad hunc casum accommodabitur, si ob punctum h in A incidens ponatur $h = 0$ et $\sqrt{(bb - hh)} = +b$, quo facto habebimus

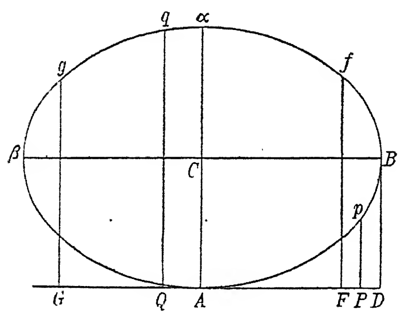

Fig. 9.

$$g = \frac{2bbf\sqrt{(bb-ff)(bb-nff)}}{b^4-nf^4} \quad \text{et} \quad g = -f,$$

sicque erit $AG = AF = f$ et ternae partes ellipsis ita different, ut sit

$$f\alpha g - ABf = \frac{nf^3}{bb} \quad \text{et} \quad ABf - A\beta g = 0.$$

Cum autem sit $g = -f$, erit

$$2bbf\sqrt{(bb - ff)(bb - nff)} = -(b^4 - nf^4)f,$$

unde quadratis sumtis elicitur

$$nnf^8 - 6nb^4f^4 + 4(n+1)b^6ff - 3b^8 = 0.$$

Ad hanc aequationem resolvendam fingantur eius factores

$$(nf^4 + Pff + Q)(nf^4 - Pff + R) = 0$$

esseque oportet

$$-6nb^4 = n(Q + R) - PP, \quad 4(n+1)b^6 = P(R - Q), \quad -3b^8 = QR,$$

ex quibus fit

$$R + Q = \frac{PP - 6nb^4}{n}, \quad R - Q = \frac{4(n+1)b^6}{P},$$

unde valores ipsarum Q et R in postrema aequatione substituta praebent

$$P^6 - 12nb^4P^4 + 48nnb^8P^2 = 16nn(n+1)^2b^{12},$$

ubi commode evenit, ut subtrahendo utrinque $64n^3b^{12}$ cubus relinquatur, cuius radice extracta fiet

$$PP - 4nb^4 = 2b^4\sqrt[3]{2nn(1-n)^2} \quad \text{et} \quad P = bb\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})}.$$

Quo valore substituto reperietur

$$R + Q = \frac{-2b^4(n - \sqrt[3]{2nn(1-n)^2})}{n},$$

$$R - Q = \frac{2b^4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})}}{n}.$$

Deinde vero ipsa resolutio suppeditat

$$ff = \frac{-P \pm \sqrt{(PP - 4nQ)}}{2n} \quad \text{et} \quad ff = \frac{+P \pm \sqrt{(PP - 4nR)}}{2n},$$

unde substitutis valoribus inventis obtinebitur

$$\frac{2nff}{bb} = -\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})} \pm \sqrt{(8n - 2\sqrt[3]{2nn(1-n)^2}}$$
$$+ 4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})}),$$

$$\frac{2nff}{bb} = +\sqrt{(4n + 2\sqrt[3]{2nn(1-n)^2})} \pm \sqrt{(8n - 2\sqrt[3]{2nn(1-n)^2}}$$
$$- 4\sqrt{(4nn - 2n\sqrt[3]{2nn(1-n)^2} + \sqrt[3]{4n^4(1-n)^4})});$$

ex his autem quaternis valoribus alii locum habere nequeunt, nisi qui ff praebeant positivum et minus quam bb.

Invento iam valore idoneo pro f pro punctis quaesitis p et q ponantur abscissae $AP = p$ et $AQ = q$ ac statuatur

$$0 = b^4(pp + qq - ff) - 2bbpq\sqrt{(bb - ff)(bb - nff)} - nffppqq$$

eritque

$$Af - pq = \frac{nfpq}{bb}$$

hincque

$$3Af - 3pq = \frac{3nfpq}{bb}.$$

Supra autem habebamus

$$fg - Af = \frac{nf^3}{bb}, \quad Ag - Af = 0,$$

quae tres aequationes additae dant

$$Af + fg + gA - 3pq = \frac{3nfpq + nf^3}{bb}.$$

Quare ut arcus pq praecise sit triens totius peripheriae, necesse est, ut sit

$$3pq = -ff \quad \text{seu} \quad pq = -\frac{1}{3}ff,$$

unde fit

$$pp + qq = ff - \frac{2ff}{3bb}\sqrt{(bb-ff)(bb-nff)} + \frac{nf^6}{9b^4}$$

hincque porro

$$qq \pm 2pq + pp = ff \pm \frac{2}{3}ff - \frac{2ff}{3bb}\sqrt{(bb-ff)(bb-nff)} + \frac{nf^6}{9b^4}.$$

Fiet ergo

$$q - p = \frac{f}{3bb}\sqrt{(15b^4 + nf^4 - 6bb\sqrt{(bb-ff)(bb-nff)})},$$

$$q + p = \frac{f}{3bb}\sqrt{(\,3b^4 + nf^4 - 6bb\sqrt{(bb-ff)(bb-nff)})}.$$

Quia rectangulum $pq = -\frac{1}{3}ff$ est negativum, patet binarum abscissarum p et q alteram esse positivam, alteram negativam. Cum autem singulis abscissis bina curvae puncta respondeant, utrum conveniat, ex valoribus $\sqrt{(bb-pp)}$ et $\sqrt{(bb-qq)}$, sive sint positivi sive negativi, dignoscitur. Eorum autem signa ita comparata esse oportet, ut satisfiat huic formulae

$$\sqrt{(bb-qq)} = \frac{b^3\sqrt{(bb-ff)(bb-pp)} - bfp\sqrt{(bb-nff)(bb-npp)}}{b^4 - nffpp}.$$

CASUS $n = \frac{1}{2}$

57. Prae ceteris hic casus $n = \frac{1}{2}$ seu $bb = 2aa$ est notatu dignus, quod hoc solo radicale cubicum rationale evadit. Erit scilicet

$$\sqrt[3]{2nn(1-n)^2} = \frac{1}{2} \quad \text{et} \quad P = bb\sqrt{3};$$

unde

$$R + Q = 0 \quad \text{et} \quad R - Q = 2b^4\sqrt{3}$$

ideoque

$$Q = -b^4\sqrt{3} \quad \text{et} \quad R = +b^4\sqrt{3}.$$

Cum iam sit

$$ff = -P \pm \sqrt{(PP - 2Q)} \quad \text{et} \quad ff = +P \pm \sqrt{(PP - 2R)},$$

erit

$$\frac{ff}{bb} = -\sqrt{3} \pm \sqrt{(3 + 2\sqrt{3})} \quad \text{et} \quad \frac{ff}{bb} = +\sqrt{3} \pm \sqrt{(3 - 2\sqrt{3})}.$$

Horum quatuor valorum bini posteriores sunt imaginarii, priorum vero solus positivus locum habet, ita ut sit

$$ff = bb\left(-\sqrt{3} + \sqrt{(3 + 2\sqrt{3})}\right),$$

quia hinc $ff < bb$. Cum porro punctum f supra axem ellipsis CB existat, erit

$$\sqrt{(bb - ff)} = -b\sqrt{\left(1 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})}\right)}$$

et

$$\sqrt{(bb - nff)} = \frac{b}{\sqrt{2}}\sqrt{\left(2 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})}\right)},$$

unde

$$\sqrt{(bb - ff)(bb - nff)} = \frac{-bb}{\sqrt{2}}\sqrt{\left(8 + 5\sqrt{3} - (3 + 2\sqrt{3})\sqrt{(3 + 2\sqrt{3})}\right)}$$

sive

$$\sqrt{(bb - ff)(bb - nff)} = -\frac{1}{2}bb\left(\sqrt{(9 + 6\sqrt{3})} - 2 - \sqrt{3}\right).$$

Cum nunc sit

$$ff = bb\left(\sqrt{(3 + 2\sqrt{3})} - \sqrt{3}\right),$$

erit

$$2pq = -\frac{2}{3}bb\left(\sqrt{(3 + 2\sqrt{3})} - \sqrt{3}\right)$$

et

$$pp + qq = +\frac{2}{3}bb\left(3 - \frac{1}{3}\sqrt{(9 + 6\sqrt{3})}\right),$$

ex quibus fit

$$(q + p)^2 = \frac{2}{3}bb\left(+3 + \sqrt{3} - \sqrt{(3 + 2\sqrt{3})} - \frac{1}{3}\sqrt{(9 + 6\sqrt{3})}\right),$$

$$(q - p)^2 = \frac{2}{3}bb\left(+3 - \sqrt{3} + \sqrt{(3 + 2\sqrt{3})} - \frac{1}{3}\sqrt{(9 + 6\sqrt{3})}\right)$$

et radicibus extractis

$$q + p = \frac{1}{3}b\sqrt{(3 + \sqrt{3})(6 - 2\sqrt{(3 + 2\sqrt{3})})},$$

$$q - p = \frac{1}{3}b\sqrt{(3 - \sqrt{3})(6 + 2\sqrt{(3 + 2\sqrt{3})})}.$$

Hinc in fractionibus decimalibus erit

$$ff = 0{,}8104090\,bb, \qquad f = 0{,}9002272\,b,$$
$$\sqrt{(bb-ff)} = -0{,}4354205\,b, \qquad \sqrt{(bb-nff)} = +0{,}7712300\,b,$$
$$2pq = -0{,}5402727\,bb, \qquad (q+p)^2 = 0{,}4811342\,bb,$$
$$pp+qq = +1{,}0214069\,bb, \qquad (q-p)^2 = 1{,}5616796\,bb,$$
$$q+p = 0{,}6936383\,b, \qquad p = 0{,}9716548\,b,$$
$$p-q = 1{,}2496712\,b, \qquad q = -0{,}2780165\,b,$$

quos valores pro p et q figura propemodum refert; atque ex formula

$$\sqrt{(bb-pp)} \quad \text{et} \quad \sqrt{(bb-qq)}$$

involvente intelligitur punctum p infra axem βB, punctum q vero supra eum capi debere.

CONSIDERATIO FORMULARUM QUARUM INTEGRATIO PER ARCUS SECTIONUM CONICARUM ABSOLVI POTEST

Commentatio 273 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae 8 (1760/1), 1763, p. 129—149
Summarum ibidem p. 21—23

SUMMARIUM

Quando integrationes algebraice perficere non licet, valores integralium per quadraturas linearum curvarum vulgo exhiberi solent, dum scilicet linea curva assignatur, cuius area eundem valorem exprimat vel saltem eiusmodi quantitatem, ex qua is determinari possit. Inter huiusmodi quantitates, quae, dum limites Algebrae communis quasi transcendunt, *Transcendentes* appellantur, frequentissime occurrunt, quae a quadratura circuli et hyperbolae pendent, quorsum omnes formulas integrales nullam irrationalitatem involventes reduci posse constat, atque hae binae transcendentium species iam ita usu in Analysin sunt receptae, ut propemodum instar algebraicarum tractentur. Quae nimirum a quadratura circuli pendent, eae nunc quidem per calculum angulorum felicissime expediuntur, quemadmodum eae, quae a quadratura hyperbolae pendent, logarithmis comprehendi solent, quorum calculus nunc fere inter elementa refertur. Quodsi vero quadraturis magis complicatis opus est, evolutio multo maioribus difficultatibus est obnoxia. Etsi enim descriptio linearum curvarum conceditur, tamen in praxi nimis est molestum areas iis inclusas satis exacte dimetiri. Quam ob causam iam pridem geometrae in hoc elaboraverunt, ut loco quadraturarum potius rectificationes curvarum in hunc usum traducerent; quia, statim ac linea curva accurate fuerit descripta, longitudinem cuiusque arcus sine ullo apparatu ope fili dimetiri licet, in quo negotio olim HERMANNUS[1]) immortalem gloriam est assecutus, dum problema ab aliis pro

1) IAC. HERMANN, *Solutio propria duorum problematum geometricorum in Actis Erudit. 1719 Mens. Aug. a se propositorum*, Acta erud. 1723, p. 171. A. K.

desperato habitum summa sagacitate resolvit et pro cuiuscunque curvae quadratura lineas curvas adeo algebraicas invenire docuit, quarum rectificatione idem praestari queat. Cum igitur nullum sit dubium, quin huiusmodi constructiones eo sint elegantiores, quo facilius curvae, quarum rectificatio adhibetur, describi queant, in hoc negotio sectionibus conicis, Ellipsi scilicet et Hyperbolae, merito primae partes sunt tribuendae; et cum plerumque difficillimum sit indolem earum formularum integralium perspicere, quarum valores per arcus sive ellipticos sive hyperbolicos exprimere liceat, Auctor hic singulari methodo praecipuas formulas integrales investigat, quae hoc modo constructionem admittunt. Celeb. ALEMBERTUS[1]) quidem hoc idem argumentum iam pridem in Actis Acad. Reg. Prussicae pertractavit, EULERI vero methodus plane nova, qua arcus sectionum conicarum aliarumque curvarum inter se comparare docuit, in hac investigatione eximiam praestitit utilitatem, ut hoc negotium multo uberius confecisse videatur. Plurimae autem transformationes, quibus Auctor in hac ardua evolutione utitur, in Analysi haud spernendam utilitatem habere possunt. Interim laudi ac dignitati huiusmodi investigationum nihil detrahetur, si observaverimus nunc quidem in calculi applicatione ad praxin neque curvarum quadraturam neque rectificationem magnopere desiderari, cum omnia multo facilius et accuratius per methodos appropinquandi expediri queant.

LEMMATA[2])

$$\text{I.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{1}{k}\int dx\sqrt{\frac{fk-gh+gxx}{xx-h}}$$

posito $x=\sqrt{(h+kzz)}$.

$$\text{II.}\quad \int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{1}{g}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}}=\frac{1}{k}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$$

posito $x=\sqrt{(f+gzz)}$ et $y=\sqrt{(h+kzz)}$.

1) I. D'ALEMBERT, *Recherches sur le calcul intégral. Seconde partie: Des différentielles qui se rapportent à la rectification de l'ellipse ou de l'hyperbole*, Mém. de l'acad. d. sc. de Berlin, (1746), 1748, p. 200; *Suite des recherches sur le calcul intégral,* Mém de l'acad. d. sc. de Berlin, 4 (1748), 1750, p. 249. A. K.

2) Demonstrationes lemmatum et theorematum sequentium reperiuntur in Commentatione 295 (indicis ENESTROEMIANI), quae sine dubio est hac prior; vide p. 256. A. K.

$$\text{III.}\quad \int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}}=-\frac{1}{k}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}}=\frac{1}{h}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

$$\text{posito } x=\frac{1}{\sqrt{(h+kzz)}}\quad\text{et}\quad y=\frac{z}{\sqrt{(h+kzz)}}.$$

$$\text{IV.}\quad \int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}=-\frac{1}{g}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}}=\frac{1}{f}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

$$\text{posito } x=\frac{1}{\sqrt{(f+gzz)}}\quad\text{et}\quad y=\frac{z}{\sqrt{(f+gzz)}}.$$

$$\text{V.}\quad \int\frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=\frac{1}{f}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}}=\frac{1}{fk-gh}\int dy\sqrt{\frac{k-gyy}{fyy-h}}$$

$$\text{posito } x=\frac{z}{\sqrt{(f+gzz)}}\quad\text{et}\quad y=\sqrt{\frac{h+kzz}{f+gzz}}.$$

$$\text{VI.}\quad \int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=\frac{1}{h}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}}=\frac{1}{gh-fk}\int dy\sqrt{\frac{g-kyy}{hyy-f}}$$

$$\text{posito } x=\frac{z}{\sqrt{(h+kzz)}}\quad\text{et}\quad y=\sqrt{\frac{f+gzz}{h+kzz}}.$$

$$\text{VII.}\quad \int\frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=-\frac{1}{g}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}}=\frac{1}{fk-gh}\int dy\sqrt{\frac{fyy-h}{k-gyy}}$$

$$\text{posito } x=\frac{1}{\sqrt{(f+gzz)}}\quad\text{et}\quad y=\sqrt{\frac{h+kzz}{f+gzz}}.$$

$$\text{VIII.}\quad \int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=-\frac{1}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}}=\frac{1}{gh-fk}\int dy\sqrt{\frac{hyy-f}{g-kyy}}$$

$$\text{posito } x=\frac{1}{\sqrt{(h+kzz)}}\quad\text{et}\quad y=\sqrt{\frac{f+gzz}{h+kzz}}.$$

THEOREMATA

$$\text{I.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{1}{k}\int dx\sqrt{\frac{fk-gh+gxx}{xx-h}}$$

$$\text{posito } x=\sqrt{(h+kzz)}.$$

$$\text{II.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=z\sqrt{\frac{f+gzz}{h+kzz}}-\int dx\sqrt{\frac{hxx-f}{g-kxx}}$$

posito $x=\sqrt{\frac{f+gzz}{h+kzz}}$.

$$\text{III.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=z\sqrt{\frac{f+gzz}{h+kzz}}+\frac{gh-fk}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$$

posito $x=\frac{1}{\sqrt{(h+kzz)}}$.

$$\text{IV.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}}+\frac{fk-gh}{k}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

posito $x=\frac{z}{\sqrt{(f+gzz)}}$.

$$\text{V.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{g}{k}z\sqrt{\frac{h+kzz}{f+gzz}}+\frac{f}{k}\int dx\sqrt{\frac{k-gxx}{fxx-h}}$$

posito $x=\sqrt{\frac{h+kzz}{f+gzz}}$.

$$\text{VI.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}}+\frac{gh-fk}{gh}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}}$$

posito $x=\sqrt{(f+gzz)}$.

$$\text{VII.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=\frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}}+\frac{gh-fk}{hk}\int dx\sqrt{\frac{xx-h}{fk-gh+gxx}}$$

posito $x=\sqrt{(h+kzz)}$.

$$\text{VIII.}\quad \int dz\sqrt{\frac{f+gzz}{h+kzz}}=z\sqrt{\frac{f+gzz}{h+kzz}}+P+Q,$$

ubi

$$P=\frac{gh-fk}{gk}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}}=\frac{fk-gh}{gh}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

posito $x=\frac{1}{\sqrt{(h+kzz)}}$ et $y=\frac{z}{\sqrt{(h+kzz)}}$

et

$$Q=\frac{-f(fk-gh)}{gh}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}}=\frac{f}{g}\int dy\sqrt{\frac{g-kyy}{kyy-f}}$$

posito $x=\frac{z}{\sqrt{(h+kzz)}}$ et $y=\sqrt{\frac{f+gzz}{h+kzz}}$.

$$\text{IX. } \int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{fk}{gh} z \sqrt{\frac{f+gzz}{h+kzz}} + P + Q,$$

ubi

$$P = \frac{gh-fk}{gh} \int dx \sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gh-fk}{hk} \int dy \sqrt{\frac{yy-h}{fk-gh+gyy}}$$

$$\text{posito } x = \sqrt{(f+gzz)} \quad \text{et} \quad y = \sqrt{(h+kzz)}$$

atque

$$Q = \frac{f(gh-fk)}{gh} \int dx \sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{f}{g} \int dy \sqrt{\frac{g-kyy}{hyy-f}}$$

$$\text{posito } x = \frac{z}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \sqrt{\frac{f+gzz}{h+kzz}}.$$

$$\text{X. } \int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{gh-fk}{gh} z \sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{h} \int dz \sqrt{\frac{h+kzz}{f+gzz}} + P,$$

ubi

$$P = \frac{gh-fk}{gk} \int dx \sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{fk-gh}{gh} \int dy \sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \frac{z}{\sqrt{(h+kzz)}}.$$

$$\text{XI. } \int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{f}{h} z \sqrt{\frac{h+kzz}{f+gzz}} + P + Q,$$

ubi

$$P = \frac{gh-fk}{gh} \int dx \sqrt{\frac{xx-f}{gh-fk+kxx}} = \frac{gh-fk}{hk} \int dy \sqrt{\frac{yy-h}{fk-gh+gyy}}$$

$$\text{posito } x = \sqrt{(f+gzz)} \quad \text{et} \quad y = \sqrt{(h+kzz)}$$

atque

$$Q = \frac{f(fk-gh)}{gh} \int dx \sqrt{\frac{1-fxx}{k+(gh-fk)xx}} = \frac{-f}{h} \int dy \sqrt{\frac{fyy-h}{k-gyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{h+kzz}{f+gzz}}.$$

$$\text{XII. } \int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{g}{k} z \sqrt{\frac{h+kzz}{f+gzz}} + P + Q,$$

ubi

$$P = \frac{f(gh-fk)}{ghk} \int dx \sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{fk-gh}{hk} \int dy \sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{z}{f+gzz}}$$

atque

$$Q = \frac{f(fk-gh)}{gh}\int dx \sqrt{\frac{1-fxx}{k+(gh-fk)xx}} = \frac{-f}{h}\int dy \sqrt{\frac{fyy-h}{k-gyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{h+kzz}{f+gzz}}.$$

$$\text{XIII.} \quad \int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{gh-fk}{hk} z \sqrt{\frac{h+kzz}{f+gzz}} + \frac{f}{h}\int dz \sqrt{\frac{h+kzz}{f+gzz}} + P,$$

ubi

$$P = \frac{f(gh-fk)}{ghk}\int dx \sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{fk-gh}{hk}\int dy \sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

$$\text{posito } x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \frac{z}{\sqrt{(f+gzz)}}.$$

THEOREMA SINGULARE[1]

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \frac{-gxz}{\sqrt{p}} - \int dx \sqrt{\frac{f+gxx}{h+kxx}},$$

ubi p denotat constantem arbitrariam, posita inter x et z hac relatione

$$gkxxzz - pxx - pzz - 2xz\sqrt{(p+fk)(p+gh)} + fh = 0$$

sive

$$x = \frac{-z\sqrt{(p+fk)(p+gh)} + \sqrt{p(f+gzz)(h+kzz)}}{p-gkzz}.$$

HYPOTHESIS

Haec scribendi formula $\Pi x\,[a]$ denotet sectionis conicae, cuius semiparameter $=1$ et semiaxis transversus $=a$, arcum a vertice sumtum, cui in axe transverso conveniat abscissa $=x$.

COROLLARIUM

Si a sit quantitas positiva, hoc modo designatur arcus ellipsis, sin negativa, arcus hyperbolae, si modo x fuerit quantitas positiva et minor quam $2a$.

1) Confer Commentationes 261 et 264 (indicis ENESTROEMIANI); vide p. 153 et 201. A. K.

INTEGRATIONES FORMULAE $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ IN 12 CASUS DISTRIBUTAE

Casus I $\int dz \sqrt{\frac{f+gzz}{h-kzz}}$

Integrale est immediate

$$C-\frac{fk+gh}{k\sqrt{fk}}\,\Pi\frac{fk}{fk+gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk+gh}\right]$$

vel etiam per theor. I

$$C+\frac{f}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{fk+gh}{fk}\right].$$

Casus II $\int dz \sqrt{\frac{f-gzz}{h-kzz}}$ existente $fk>gh$

Integrale est immediate

$$C-\frac{fk-gh}{k\sqrt{fk}}\,\Pi\frac{fk}{fk-gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk-gh}\right]$$

vel etiam per theor. I

$$C+\frac{f}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{fk-gh}{fk}\right].$$

Casus III $\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$ existente $fk<gh$

Integrale est immediate

$$C+\frac{gh-fk}{k\sqrt{fk}}\,\Pi\frac{fk}{gh-fk}\left(z\sqrt{\frac{k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right].$$

Casus IV $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ existente $fk<gh$

Integrale est per theor. I

$$C+\frac{f}{\sqrt{(gh-fk)}}\,\Pi\frac{gh-fk}{fk}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-gh+fk}{fk}\right].$$

Casus V $\int dz \sqrt{\frac{-f+gzz}{h+kzz}}$

Integrale est per theor. III

$$C+z\sqrt{\frac{-f+gzz}{h+kzz}}-\frac{f}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{fk}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk+gh}{fk}\right]$$

vel etiam per theor. II

$$C+z\sqrt{\frac{-f+gzz}{h+kzz}}+\frac{fk+gh}{k\sqrt{fk}}\,\varPi\frac{fk}{fk+gh}\left(1-\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk}{fk+hg}\right].$$

Casus VI $\int dz\sqrt{\frac{-f+gzz}{-h+kzz}}$ existente $fk>gh$

Integrale est per theor. III

$$C+z\sqrt{\frac{-f+gzz}{-h+kzz}}-\frac{f}{\sqrt{(fk-gh)}}\,\varPi\frac{fk-gh}{fk}\left(1-\frac{\sqrt{(fk-gh)}}{\sqrt{g(-h+kzz)}}\right)\left[\frac{fk-gh}{fk}\right]$$

vel etiam per theor. II

$$C+z\sqrt{\frac{-f+gzz}{-h+kzz}}+\frac{fk-gh}{k\sqrt{fk}}\,\varPi\frac{fk}{fk-gh}\left(1-\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-h+kzz)}}\right)\left[\frac{fk}{fk-gh}\right].$$

Casus VII $\int dz\sqrt{\frac{f-gzz}{h-kzz}}$ existente $fk<gh$

Integrale est per theor. III

$$C+\frac{gz}{k}\sqrt{\frac{h-kzz}{f-gzz}}-\frac{gh-fk}{k\sqrt{fk}}\,\varPi\frac{fk}{gh-fk}\left(\frac{\sqrt{f(h-kzz)}}{\sqrt{h(f-gzz)}}-1\right)\left[\frac{-fk}{gh-fk}\right].$$

Casus VIII $\int dz\sqrt{\frac{-f+gzz}{h-kzz}}$ existente $fk<gh$

Integrale est per theor. II

$$C+z\sqrt{\frac{-f+gzz}{h-kzz}}-\frac{f}{\sqrt{(gh-fk)}}\,\varPi\frac{gh-fk}{fk}\left(\frac{\sqrt{(gh-fk)}}{\sqrt{g(h-kzz)}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

vel etiam per theor. V

$$C-\frac{gz}{k}\sqrt{\frac{h-kzz}{-f+gzz}}+\frac{f}{\sqrt{(gh-fk)}}\,\varPi\frac{gh-fk}{fk}\left(\frac{z\sqrt{(gh-fk)}}{\sqrt{h(-f+gzz)}}-1\right)\left[\frac{-gh+fk}{fk}\right].$$

Casus IX $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ existente $fk>gh$

Integrale est per theor. X

$$C-\frac{(fk-gh)z}{gh}\sqrt{\frac{f+gzz}{h+kzz}}-\frac{fk-gh}{k\sqrt{fk}}\,\varPi\frac{fk}{gh}\left(1-\frac{z\sqrt{k}}{\sqrt{(h+kzz)}}\right)\left[\frac{fk}{gh}\right]$$
$$+\frac{f}{\sqrt{(fk-gh)}}\,\varPi\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$

vel etiam per theor. XIII

$$C - \frac{(fk-gh)z}{hk}\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(1-\frac{\sqrt{f}}{\sqrt{(f+gzz)}}\right)\left[\frac{fk}{gh}\right]$$
$$+\frac{f}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right]^{1)}.$$

Casus X $\int dz\sqrt{\frac{f-gzz}{-h+kzz}}$ existente $fk>gh$

Integrale est per theor. IX

$$C+\frac{fkz}{gh}\sqrt{\frac{f-gzz}{-h+kzz}}+\frac{fk-gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(1-\frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gh)}}\right)\left[\frac{fk}{gh}\right]$$
$$-\frac{f}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{z\sqrt{(fk-gh)}}{\sqrt{f(-h+kzz)}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$

vel etiam per theor. XI

$$C-\frac{fz}{h}\sqrt{\frac{-h+kzz}{f-gzz}}+\frac{fk-gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(1-\frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gh)}}\right)\left[\frac{fk}{gh}\right]$$
$$+\frac{f}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{\sqrt{(fk-gh)}}{\sqrt{k(f-gzz)}}-1\right)\left[\frac{-fk+gh}{gh}\right].$$

Casus XI $\int dz\sqrt{\frac{f+gzz}{-h+kzz}}$

Integrale est per theor. XI

$$C-\frac{fz}{h}\sqrt{\frac{-h+kzz}{f+gzz}}+\frac{f}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{gh}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{k(f+gzz)}}\right)\left[\frac{fk+gh}{gh}\right]$$
$$+\frac{fk+gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(\frac{\sqrt{k(f+gzz)}}{\sqrt{(fk+gh)}}-1\right)\left[\frac{-fk}{gh}\right]$$

vel etiam per theor. XII

$$C+\frac{gz}{k}\sqrt{\frac{-h+kzz}{f+gzz}}+\frac{f}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{gh}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{k(f+gzz)}}\right)\left[\frac{fk+gh}{gh}\right]$$
$$+\frac{fk+gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(\frac{\sqrt{f}}{\sqrt{(f+gzz)}}-1\right)\left[\frac{-fk}{gh}\right].$$

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1) Editio princeps: $+\frac{f}{\sqrt{(fh-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-fk+gh}{gh}\right].$

Correxit A. K.

Casus XII $\int dz \sqrt{\frac{f-gzz}{h+kzz}}$

Integrale est per theor. XIII

$$C-\frac{(fk+gh)}{hk}z\sqrt{\frac{h+kzz}{f-gzz}}+\frac{f}{\sqrt{(fk+gh)}}\Pi\frac{fk+gh}{gh}\left(1-\frac{\sqrt{(f-gzz)}}{\sqrt{f}}\right)\left[\frac{fk+gh}{gh}\right]$$
$$+\frac{fk+gh}{k\sqrt{fk}}\Pi\frac{fk}{gh}\left(\frac{\sqrt{f}}{\sqrt{(f-gzz)}}-1\right)\left[\frac{-fk}{gh}\right].$$

Omnes ergo casus formulae

$$\int dx \sqrt{\frac{\alpha+\beta zz}{\gamma+\delta zz}},$$

quomodocunque litterae α, β, γ, δ fuerint comparatae, per arcus sectionum conicarum integrari possunt.

Non solum igitur formulae initio commemoratae integrationem per arcus sectionum conicarum admittunt, sed etiam innumerabiles aliae, quae per substitutionem ad formam

$$\int dz \sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$$

se reduci patiuntur, cuiusmodi sunt

$$1.\ \int\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}=-\int dx\sqrt{\frac{fxx+g}{hxx+k}}=-\frac{1}{h}\int dy\sqrt{\frac{fyy-fk+gh}{yy-k}}$$

$$\text{posito } x=\frac{1}{z} \quad \text{et} \quad y=\frac{\sqrt{(h+kzz)}}{z},$$

$$2.\ \int\frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}}=-\frac{1}{f}\int dx\sqrt{\frac{xx-g}{hxx+fk-gh}}=-\frac{1}{h}\int dy\sqrt{\frac{yy-k}{fyy-fk+gh}}$$

$$\text{posito } x=\frac{\sqrt{(f+gzz)}}{z} \quad \text{et} \quad y=\frac{\sqrt{(h+kzz)}}{z},$$

$$3.\ \int\frac{dz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{k}{fk-gh}\int dz\sqrt{\frac{f+gzz}{h+kzz}}-\frac{g}{fk-gh}\int dz\sqrt{\frac{h+kzz}{f+gzz}},$$

cuius formulae reductio etiam ita instituitur

$$\int\frac{dz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{f}{fk-gh}\int dx\sqrt{\frac{k-gxx}{fxx-h}}+\frac{g}{fk-gh}\int dx\sqrt{\frac{fxx-h}{k-gxx}}$$

$$\text{posito } x=\sqrt{\frac{h+kzz}{f+gzz}}$$

vel etiam sic

$$\int \frac{dz}{\sqrt{(f+gzz)(h+kzz)}} = \int dx \sqrt{\frac{1-gxx}{h+(fk-gh)xx}} - \int dy \sqrt{\frac{1-fyy}{k+(gh-fk)yy}}$$

$$\text{posito } x = \frac{z}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \frac{1}{\sqrt{(f+gzz)}}.$$

Ponamus $zz = v$ atque obtinebimus sequentes formulas, quae pariter per arcus sectionum conicarum construi poterunt

$$1.\ \int \frac{dv\sqrt{(f+gv)}}{\sqrt{v(h+kv)}} \qquad 2.\ \int \frac{dv\sqrt{(f+gv)}}{v\sqrt{v(h+kv)}}$$

$$3.\ \int \frac{dv\sqrt{v}}{\sqrt{(f+gv)(h+kv)}} \qquad 4.\ \int \frac{dv}{\sqrt{v(f+gv)(h+kv)}}$$

$$5.\ \int \frac{dv\sqrt{(f+gv)}}{(h+kv)^{\frac{3}{2}}\sqrt{v}} \qquad 6.\ \int \frac{dv}{v\sqrt{v(f+gv)(h+kv)}}$$

$$7.\ \int \frac{dv}{(f+gv)^{\frac{3}{2}}\sqrt{v(h+kv)}} \qquad 8.\ \int \frac{dv\sqrt{v}}{(f+gv)^{\frac{3}{2}}\sqrt{(h+kv)}};$$

hae enim vicissim posito $v = zz$ ad formas praecedentes reducuntur.

Hinc patet istam formulam satis late patentem ad arcus sectionum conicarum reduci posse

$$\int \frac{(A+Bu)du}{\sqrt{(\alpha+\beta u)(\gamma+\delta u)(\varepsilon+\zeta u)}},$$

quae imprimis notari meretur. Ponatur enim $\alpha + \beta u = v$, ut sit $u = \frac{v-\alpha}{\beta}$, haecque formula transmutabitur in hanc

$$\int \frac{dv(A\beta - B\alpha + Bv)}{\beta\sqrt{v(\beta\gamma - \alpha\delta + \delta v)(\beta\varepsilon - \alpha\zeta + \zeta v)}},$$

quae ad binas formulas sub no. 3 et 4 allatas revocatur. Quare si

$$\alpha + \beta x + \gamma xx + \delta x^3$$

habeat tres factores reales, haec formula

$$\int \frac{dx(A+Bx)}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3)}}$$

modo exposito integrari poterit; semper autem unum factorem certe habet

realem. Sin autem bini sint imaginarii, formula $\alpha + \beta x + \gamma xx + \delta x^3$ ita referri potest $y(pp + 2npqy + qqyy)$ existente $nn < 1$, ut definiendum sit integrale harum formularum

$$\int \frac{Cdy}{\sqrt{y(pp+2npqy+qqyy)}} + \int \frac{Ddy\sqrt{y}}{\sqrt{(pp+2npqy+qqyy)}}.$$

Ponatur

$$\sqrt{(pp + 2npqy + qqyy)} = p + qyz$$

fietque $y = \frac{2p(z-n)}{q(1-zz)}$, qua substitutione prior formula abit in

$$\frac{C\sqrt{2}}{\sqrt{pq}} \int \frac{dz}{\sqrt{(z-n)(1-z)(1+z)}}$$

construibilem, posterior vero in hanc

$$\frac{2D\sqrt{2p}}{q\sqrt{q}} \int \frac{dz\sqrt{(z-n)}}{(1-zz)^{\frac{3}{2}}};$$

cum vero sit

$$\int \frac{dz\sqrt{(z-n)}}{(1-zz)^{\frac{3}{2}}} = \frac{z\sqrt{(z-n)}}{\sqrt{(1-zz)}} - \frac{1}{2} \int \frac{zdz}{\sqrt{(z-n)(1-z)(1+z)}},$$

etiam haec per superiora construi potest. Sicque in genere habetur constructio huius formulae

$$\int \frac{dx(A+Bx)}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3)}}.$$

PROBLEMA 1

Integrationem huius formulae

$$\int \frac{dx}{\sqrt{(a+bx+cxx+dx^3+ex^4)}}$$

per arcus sectionum conicarum perficere.

SOLUTIO

Quantitatem $a + bx + cxx + dx^3 + ex^4$ semper in duos factores trinomiales reales resolvere licet, qui sint $(\alpha + 2\beta x + \gamma xx)$ et $(\delta + 2\varepsilon x + \zeta xx)$,

ita ut habeatur haec formula integranda

$$\int \frac{dx}{\sqrt{(\alpha + 2\beta x + \gamma xx)(\delta + 2\varepsilon x + \zeta xx)}}.$$

Ponatur

$$\delta + 2\varepsilon x + \zeta xx = (\alpha + 2\beta x + \gamma xx)y,$$

ut formula proposita fiat

$$\int \frac{dx}{(\alpha + 2\beta x + \gamma xx)\sqrt{y}}.$$

At aequatio assumta per radicis extractionem praebet

$$\varepsilon + \zeta x - \beta y - \gamma xy = \sqrt{(pyy + qy + r)}$$

posito

$$p = \beta\beta - \alpha\gamma, \quad q = \alpha\zeta - 2\beta\varepsilon + \gamma\delta \quad \text{et} \quad r = \varepsilon\varepsilon - \delta\zeta.$$

Tum vero eadem differentiata dat

$$dx(\varepsilon + \zeta x - \beta y - \gamma xy) = \frac{1}{2}dy(\alpha + 2\beta x + \gamma xx)$$

seu

$$\frac{dx}{\alpha + 2\beta x + \gamma xx} = \frac{\frac{1}{2}dy}{\varepsilon + \zeta x - \beta y - \gamma xy}.$$

Quare si pro hoc postremo denominatore valorem irrationalem modo inventum substituamus, formula proposita abit in hanc

$$\int \frac{\frac{1}{2}dy}{\sqrt{y(pyy + qy + r)}},$$

cuius integratio per arcus sectionum conicarum supra est ostensa.

Hic igitur nascitur quaestio, quid tenendum sit de hac formula

$$\int \frac{dx(A + Bx + Cxx)}{\sqrt{(a + bx + cxx + dx^3 + ex^4)}}.$$

Evidens enim est non necesse esse, ut numeratori altiores potestates ipsius x tribuantur; quam etiam Cel. D'ALEMBERT[1]) fatetur se in genere ad rectificationem sectionum conicarum perducere non posse. Considerat quidem in

1) Vide notam 1 p. 236. A. K.

Vol. IV Mem. Acad. R. Berol. pag. 254 casum, quo $A=0$, $C=0$ et $a=0$, ita ut formula sit

$$\int\frac{dx\sqrt{x}}{\sqrt{(b+cx+dx^2+ex^3)}},$$

conaturque ostendere (p. 257) eius integrationem casu $dd=4ce$ per arcus sectionum conicarum absolvi posse; verum methodus, qua utitur, negotium minime conficere videtur, uti rem accuratius perpendenti mox patebit. Transformationes autem, quas deinceps tradit, casus nonnunquam hoc modo tractabiles suppeditant. Quocirca haec investigatio, uti est difficillima, merito omni attentione digna est censenda, unde etiam mea tentamina super hac quaestione proposuisse iuvabit.

PROBLEMA 2

Investigare conditiones, sub quibus integrationem huius formulae

$$\int\frac{dy(\mathfrak{P}+\mathfrak{Q}y+\mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}}$$

ad hanc simpliciorem

$$\int\frac{dx(P+Qx+Rxx)}{\sqrt{(Ax^4+Cxx+E)}}$$

reducere liceat.

SOLUTIO

Statuatur inter variabiles x et y talis relatio

$$\alpha xxyy+2xy(\beta x+\gamma y)+\delta xx+\varepsilon yy+2\zeta xy+2\eta x+2\theta y+\varkappa=0,$$

cuius coefficientes ita determinentur, ut sit

$$\begin{aligned}
&\beta\zeta-\alpha\eta-\gamma\delta=0, &&\zeta\theta-\gamma\varkappa-\varepsilon\eta=0,\\
&\gamma\gamma-\alpha\varepsilon=\mathfrak{A}, &&\gamma\zeta-\alpha\theta-\beta\varepsilon=\mathfrak{B},\\
&\eta\eta-\delta\varkappa=\mathfrak{E}, &&\zeta\eta-\beta\varkappa-\delta\theta=\mathfrak{D}
\end{aligned}$$

et

$$\zeta\zeta+2\gamma\eta-\alpha\varkappa-\delta\varepsilon-4\beta\theta=\mathfrak{C},$$

hincque erit pro denominatore transformatae

$$A=\beta\beta-\alpha\delta,\quad E=\theta\theta-\varepsilon\varkappa$$

et

$$C=\zeta\zeta+2\beta\theta-\alpha\varkappa-\delta\varepsilon-4\gamma\eta.$$

Cum autem novem habeantur litterae α, β, γ, δ, ε, ζ, η, θ, $\varkappa$, his septem conditionibus praescriptis utique satisfieri poterit relinqueturque adhuc una arbitrio nostro determinanda. Si iam brevitatis gratia ponamus

$$\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E} = Y \quad \text{et} \quad Ax^4 + Cxx + E = X,$$

resolutio aequationis assumtae praebet

$$\alpha xyy + 2\beta xy + \delta x + \gamma yy + \zeta y + \eta = \sqrt{Y},$$
$$\alpha xxy + 2\gamma xy + \varepsilon y + \beta xx + \zeta x + \theta = \sqrt{X}$$

eiusque differentiatio ducit ad hanc aequationem

$$\frac{dy}{\sqrt{Y}} + \frac{dx}{\sqrt{X}} = 0.$$

Ponamus ergo

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = V - \int \frac{dx(P + Qx + Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}}$$

ac sit V talis functio algebraica

$$V = mx + ny + pxy + \frac{1}{2}qxx + \frac{1}{2}ryy + txyy.$$

Hinc sumtis differentialibus terminisque homogeneis seorsim aequatis reperientur sequentes determinationes

$$m = \frac{\beta\mathfrak{R}}{\mathfrak{A}}, \quad n = \frac{\gamma\mathfrak{R}}{\mathfrak{A}}, \quad p = \frac{\alpha\mathfrak{R}}{\mathfrak{A}}, \quad q = 0, \quad r = 0 \quad \text{et} \quad t = 0,$$

praeterea vero haec determinatio accedit, ut sit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$. Deinde vero fit

$$P = \mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}}, \quad Q = 0 \quad \text{et} \quad R = \frac{A\mathfrak{R}}{\mathfrak{A}}.$$

Definitis ergo coefficientibus α, β, γ, δ, ε, ζ, η, θ, $\varkappa$, quibus constat relatio inter x et y, ex iis innotescunt quantitates A, C, E, quibus inventis, si fuerit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$, erit

$$\frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}} = \text{Const.} + \frac{\mathfrak{R}}{\mathfrak{A}}(\beta x + \gamma y + \alpha xy)$$
$$- \int \frac{dx\left(\mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}} + \frac{A\mathfrak{R}}{\mathfrak{A}}xx\right)}{\sqrt{(Ax^4 + Cx^2 + E)}}.$$

Dummodo ergo fuerit $\mathfrak{Q}=\frac{\mathfrak{B}\mathfrak{R}}{\mathfrak{A}}$, formulae propositae integratio reducta est ad hanc simpliciorem

$$\int\frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cx^2+E)}}.$$

COROLLARIUM 1

Determinatio coefficientium α, β, γ etc. commodissime hoc modo instituetur. Primo quaeratur valor ipsius s ex hac aequatione

$$\mathfrak{C}=\frac{\mathfrak{B}\mathfrak{B}-\mathfrak{D}\mathfrak{D}ss}{\mathfrak{A}-\mathfrak{E}ss}+\frac{2\mathfrak{A}\mathfrak{D}-2\mathfrak{B}\mathfrak{E}s}{\mathfrak{B}-\mathfrak{D}s},$$

quae cum sit cubica, certe valorem realem pro s suggerit; quo invento sumtaque ad arbitrium quantitate t sit brevitatis gratia $\frac{\mathfrak{A}-\mathfrak{E}ss}{\mathfrak{B}-\mathfrak{D}s}=u$, tum autem valores omnium 9 coefficientium ita se habebunt

$$\zeta=u\sqrt{\frac{\mathfrak{A}\mathfrak{B}-3\mathfrak{A}\mathfrak{D}s+3\mathfrak{B}\mathfrak{E}ss-\mathfrak{D}\mathfrak{E}s^3}{s(s-uu)(\mathfrak{B}-\mathfrak{D}s)}}\,{}^{1)},$$

$$\gamma=\frac{\zeta s}{2u},\quad \alpha=\frac{u}{2t(s-uu)},$$

$$\eta=\frac{\zeta}{2u},\quad \delta=\frac{u}{2st(s-uu)},$$

$$\beta=\frac{1}{2t(s-uu)},\quad \theta=\frac{1}{2}t\Big(2(\mathfrak{A}+\mathfrak{E}ss)-\frac{3}{u}(\mathfrak{B}+\mathfrak{D}s)\Big),$$

$$\varepsilon=\frac{1}{2}t(4\mathfrak{A}u-3\mathfrak{B}s+\mathfrak{D}ss),\quad \varkappa=\frac{1}{2}t(4\mathfrak{E}su+\mathfrak{B}-3\mathfrak{D}s).$$

COROLLARIUM 2

Alio adhuc modo idem praestari potest. Extracto scilicet ut ante valore s ex hac aequatione

$$\mathfrak{C}=\frac{\mathfrak{B}\mathfrak{B}-\mathfrak{D}\mathfrak{D}ss}{\mathfrak{A}-\mathfrak{E}ss}+\frac{2\mathfrak{A}\mathfrak{D}-2\mathfrak{B}\mathfrak{E}s}{\mathfrak{B}-\mathfrak{D}s}$$

positoque brevitatis gratia $\frac{\mathfrak{A}-\mathfrak{E}ss}{\mathfrak{B}-\mathfrak{D}s}=u$ et sumto t pro arbitrio erit

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1) Editio princeps: $\zeta=u\sqrt{\frac{\mathfrak{A}\mathfrak{B}-3\mathfrak{A}\mathfrak{D}s+3\mathfrak{B}\mathfrak{E}ss-\mathfrak{D}\mathfrak{E}s^3}{s-uu}}$. Correxit A. K.

$$\alpha = -\frac{1}{4tu}, \quad \beta = 0, \quad \gamma = \frac{1}{2}\sqrt{\frac{s(\mathfrak{B}+\mathfrak{D}s)}{u}}, \quad \delta = \frac{1}{4tsu},$$

$$\varepsilon = t(4\mathfrak{A}u - \mathfrak{B}s - \mathfrak{D}ss), \quad \zeta = \sqrt{\frac{u(\mathfrak{B}+\mathfrak{D}s)}{s}}, \quad \eta = \frac{1}{2}\sqrt{\frac{\mathfrak{B}+\mathfrak{D}s}{us}},$$

$$\theta = 2tu(\mathfrak{B} - \mathfrak{D}s)^{1)}, \quad \varkappa = t(\mathfrak{B} + \mathfrak{D}s - 4\mathfrak{E}su).$$

COROLLARIUM 3

Si fuerit $\mathfrak{A} : \mathfrak{E} = \mathfrak{B}\mathfrak{B} : \mathfrak{D}\mathfrak{D}$, aequatio cubica valori s definiendo fit inepta. Hoc autem incommodum facile tollitur transformanda formula differentiali per positionem $y = y \pm a$; qua etiam forma numeratoris non turbatur.

SCHOLION

Posito $\mathfrak{R} = n\mathfrak{A}$ et $\mathfrak{Q} = n\mathfrak{B}$ integratio huius formulae

$$\int \frac{dy(\mathfrak{P} + n\mathfrak{B}y + n\mathfrak{A}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}}$$

semper reduci potest ad integrationem talis

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^4 + Cxx + E)}},$$

quae, si denominator $Ax^4 + Cxx + E$ in huiusmodi duos factores reales $(f + gxx)(h + kxx)$ se resolvi patitur, per rectificationem sectionum conicarum conficitur; at si talis resolutio non succedit, sequenti artificio negotium absolvi poterit.

PROBLEMA 3

Si in formula

$$\int \frac{dx(P + Rxx)}{\sqrt{(Ax^4 + Cx^2 + E)}}$$

quantitas $Ax^4 + Cx^2 + E$ *in factores reales huiusmodi* $(f + gxx)(h + kxx)$ *resolvi nequeat, eam in aliam transformare, quae per arcus sectionum conicarum certo integrari queat.*

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1) Editio princeps: $\theta = 2tu$. Correxit A. K.

SOLUTIO

Inducatur alia variabilis z, cuius relatio ad x hac aequatione exprimatur

$$4Exxz^4 - 4xxzz\sqrt{AE} - 4Ezz + 2\sqrt{AE} - C = 0,$$

ubi $\sqrt{AE}$ erit utique quantitas realis, si quidem $Ax^4 + Cxx + E$ non habeat factores binomios reales. Hinc autem fiet

$$\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cx^2+E)}} = \text{Const.} + \frac{Rx}{\sqrt{A}} - \frac{2R\sqrt{E}}{A}xzz$$
$$-2\int \frac{dz\left(P - \frac{R\sqrt{E}}{\sqrt{A}} + \frac{2ER}{A}zz\right)}{\sqrt{\left(4Ez^4 + (C - 6\sqrt{AE})zz + 2A - \frac{C\sqrt{A}}{\sqrt{E}}\right)}},$$

in qua nova formula quantitas in denominatore contenta certe in duos factores binomios reales est resolubilis, cum sit

$$(C - 6\sqrt{AE})^2 > 16E\left(2A - \frac{C\sqrt{A}}{\sqrt{E}}\right),$$

propterea quod hinc sequitur

$$CC + 4C\sqrt{AE} + 4AE = (C + 2\sqrt{AE})^2 > 0.$$

ALITER

Habeat nova variabilis z ad x talem relationem

$$2Exxz^4 - Cxxzz + \frac{CC-4AE}{8E}xx - 2Ezz = 0$$

eritque

$$\int \frac{dx(P+Rxx)}{\sqrt{(Ax^4+Cxx+E)}} = \frac{CR}{2A\sqrt{E}}x - \frac{2R\sqrt{E}}{A}xzz$$
$$-2\int \frac{dz\left(P - \frac{CR}{2A} + \frac{2ER}{A}zz\right)}{\sqrt{\left(4Ez^4 - 2Czz + \frac{CC-4AE}{4E}\right)}},$$

cuius denominator pariter certe in factores reales binomios est resolubilis.

CONCLUSIO

His demonstratis manifestum est hanc formulam

$$\int \frac{dy(\mathfrak{P} + n\mathfrak{B}y + n\mathfrak{A}yy)}{\sqrt{(Ay^4 + 2\mathfrak{B}y^3 + \mathfrak{C}y^2 + 2\mathfrak{D}y + \mathfrak{E})}}$$

semper per arcus sectionum conicarum construi posse. Cum igitur denominator semper in duos factores trinomiales reales resolvi possit, haec formula ita exhiberi potest

$$\int \frac{dy(\mathfrak{P} + n(\alpha\varepsilon + \beta\delta)y + n\alpha\delta yy)}{\sqrt{(\alpha yy + 2\beta y + \gamma)(\delta yy + 2\varepsilon y + \zeta)}},$$

cuius ergo eadem datur constructio. Porro augendo vel diminuendo y quantitate constante formula nostra etiam ita repraesentari potest

$$\int \frac{dy(M + Nyy)}{\sqrt{(Ay^4 + Cyy + 2Dy + E)}}.$$

In his autem fere omnes casus, quos quidem per rectificationem sectionum conicarum integrare licet, contineri videntur. Sed in medium afferamus adhuc aliam reductionem.

PROBLEMA 4

Investigare conditiones, sub quibus integrationem huius formulae

$$\int \frac{dy(\mathfrak{P} + \mathfrak{Q}y + \mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4 + 2\mathfrak{B}y^3 + \mathfrak{C}yy + 2\mathfrak{D}y + \mathfrak{E})}}$$

ad hanc simpliciorem

$$\int \frac{dx(P + Qx + Rxx)}{\sqrt{(2Bx^3 + Cx^2 + 2Dx)}}$$

perducere liceat.

SOLUTIO

Statuatur inter variabiles x et y talis relatio

$$\alpha xxyy + 2xy(\beta x + \gamma y) + \delta xx + \varepsilon yy + 2\zeta xy + 2\eta x + 2\theta y + \varkappa = 0,$$

cuius coefficientes ita determinentur, ut sit

$$\beta\beta-\alpha\delta=0,\quad \gamma\gamma-\alpha\varepsilon=\mathfrak{A},\quad \gamma\zeta-\alpha\theta-\beta\varepsilon=\mathfrak{B},$$
$$\theta\theta-\varepsilon\varkappa=0,\quad \eta\eta-\delta\varkappa=\mathfrak{E},\quad \zeta\eta-\beta\varkappa-\delta\theta=\mathfrak{D}$$

atque

$$\zeta\zeta+2\gamma\eta-\alpha\varkappa-\delta\varepsilon-4\beta\theta=\mathfrak{C},$$

quem in finem definiatur primo p ex hac aequatione cubica

$$p^3-\frac{1}{2}\mathfrak{C}pp-(\mathfrak{A}\mathfrak{E}-\mathfrak{B}\mathfrak{D})p+\frac{1}{2}(\mathfrak{C}\mathfrak{A}\mathfrak{E}-\mathfrak{A}\mathfrak{D}\mathfrak{D}-\mathfrak{B}\mathfrak{B}\mathfrak{E})=0.$$

Deinde pro lubitu sumto numero m definiatur q ex hac aequatione quadratica

$$qq-q(\mathfrak{D}m-\mathfrak{B})+(m\mathfrak{E}-p)(mp-\mathfrak{A})=0,$$

quo facto, si denuo numerus arbitrarius accipiatur n, erit

$$\beta=\frac{n(m\mathfrak{E}-p)}{\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},\qquad \theta=\frac{mp-\mathfrak{A}}{n\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},$$
$$\alpha=\frac{nq}{\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},\qquad \varkappa=\frac{q}{n\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},$$
$$\delta=\frac{n(m\mathfrak{E}-p)^2}{q\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},\qquad \varepsilon=\frac{(mp-\mathfrak{A})^2}{nq\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},$$
$$\gamma=\frac{m\sqrt{(pp-\mathfrak{A}\mathfrak{E})}}{\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},\qquad \eta=\frac{\sqrt{(pp-\mathfrak{A}\mathfrak{E})}}{\sqrt{(2mp-\mathfrak{A}-mm\mathfrak{E})}},$$

et

$$\zeta=\frac{\mathfrak{D}(mp-\mathfrak{A})-\mathfrak{B}(m\mathfrak{E}-p)}{\sqrt{(pp-\mathfrak{A}\mathfrak{E})(2mp-\mathfrak{A}-mm\mathfrak{E})}}.$$

Quibus inventis erit

$$B=\beta\zeta-\alpha\eta-\gamma\delta,\quad D=\zeta\theta-\gamma\varkappa-\varepsilon\eta$$

et

$$C=\zeta\zeta+2\beta\theta-\alpha\varkappa-\delta\varepsilon-4\gamma\eta.$$

Ponatur iam

$$\int\frac{dy(\mathfrak{P}+\mathfrak{Q}y+\mathfrak{R}yy)}{\sqrt{(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}y^2+2\mathfrak{D}y+\mathfrak{E})}}=\text{Const.}+mx+ny+pxy$$
$$-\int\frac{dx(P+Qx+Rxx)}{\sqrt{(2Bx^3+Cx^2+2Dx)}}$$

atque reperitur ut ante

$$m = \frac{\beta \mathfrak{R}}{\mathfrak{A}}, \quad n = \frac{\gamma \mathfrak{R}}{\mathfrak{A}} \quad \text{et} \quad p = \frac{\alpha \mathfrak{R}}{\mathfrak{A}},$$

deinde

$$P = \mathfrak{P} + \frac{(\beta\theta - \gamma\eta)\mathfrak{R}}{\mathfrak{A}}, \quad Q = \frac{B\mathfrak{R}}{\mathfrak{A}} \quad \text{et} \quad R = 0.$$

Necesse autem est, ut in formula proposita sit $\mathfrak{A}\mathfrak{Q} = \mathfrak{B}\mathfrak{R}$, neque ergo haec reductio novos casus suppeditat. At posito $x = zz$ formula transformata abit in hanc

$$-2\int \frac{dz(P + Qzz)}{\sqrt{(2Bz^4 + Cz^2 + 2D)}},$$

quae reductio saepe facilius succedit quam praecedens.

DE REDUCTIONE FORMULARUM INTEGRALIUM AD RECTIFICATIONEM ELLIPSIS AC HYPERBOLAE

Commentatio 295 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae **10** (1764), 1766, p. 3—50
Summarium ibidem p. 5—9

SUMMARIUM

Quae quantitates numeris neque integris neque fractis neque etiam surdis vel irrationalibus exhiberi possunt, transcendentes vocari solent, quarum ergo valores non aliter nisi proxime per numeros exprimere licet. Dari autem huiusmodi quantitates certissimum est, etiamsi ratio ob infinitudinem, quae eas excludere videtur, a plerisque minus distincte perspiciatur; id quod exemplo notissimo peripheriae circuli, cuius diameter unitate indicetur, evidenter declarari potest. Nullum enim est dubium, quin quantitas huius peripheriae valorem habeat omnino determinatum, quem adeo primo intuitu constat intra limites 3 et 4 contineri. Verum intra hos limites innumerabiles constitui possunt fractiones ratione denominatorum discrepantes, cuiusmodi simpliciores sunt

$$3\tfrac{1}{2},\quad 3\tfrac{1}{3},\quad 3\tfrac{2}{3},\quad 3\tfrac{1}{4},\quad 3\tfrac{3}{4},\quad 3\tfrac{1}{5},\quad 3\tfrac{2}{5},\quad 3\tfrac{3}{5},\quad 3\tfrac{4}{5},\quad \text{etc.}$$

et generatim in hac forma $3\frac{m}{m+n}$ comprehenduntur; ubi cum tam pro m quam pro n omnes plane numeros substitui liceat, nulla tamen huiusmodi formula veram peripheriae quantitatem praebet, sed quaecunque assumatur, semper a veritate recedit, etiamsi error continuo minor reddi possit. Deinde quantitates etiam surdas introducendo multitudo numerorum intra limites 3 et 4 contentorum ulterius in infinitum augetur, qui adeo omnes ab iis, qui in formula $3\frac{m}{m+n}$ continentur, discrepant, neque tamen etiam in his ullus reperitur, qui circuli peripheriam exacte dimetiatur; quamobrem eius quantitas merito pro transcendente habetur. Quod idem multo magis de omnibus circuli arcubus est intelligendum, ita ut, quicunque capiatur sinus in circulo, arcus ipsi respondens semper sit quan-

titas transcendens sicque solus circulus infinitam quantitatum transcendentium multitudinem suppeditet. Deinde vero etiam logarithmi ad classem numerorum transcendentium sunt referendi, qui adeo ab illis, qui ex circulo nascuntur, prorsus sunt diversi. Iam nemo non videt, si praeter fractiones et quantitates transcendentes ex circulo et logarithmis ortae in subsidium vocentur, tum inter binos quosvis numeros multitudinem numerorum mediorum multo magis in immensum augeri; ex quo maxime mirum videbitur ne hoc quidem modo intervalla inter binos numeros integros ita numeris mediis expleri, ut iis omnes plane quantitates intra eosdem terminos contentae exprimi queant. Quin potius praeterea innumerabilia alia quantitatum transcendentium genera, tam inter se quam ab illis ex circulo et logarithmis natis maxime discrepantia, agnosci oportet; inter quae potissimum notari merentur ea, quae ex rectificatione ellipsium et hyperbolarum originem ducunt, propterea quod hae curvae post circulum sunt notissimae et facillime describuntur. Quomodocunque autem tam in ellipsi quam hyperbola arcus rescindantur, eorum quantitas non solum nullis formulis irrationalibus exprimi, sed etiam nullo modo neque ad arcus circulares neque ad logarithmos revocari possunt; quin etiam singuli arcus tam elliptici quam hyperbolici peculiares quantitates transcendentes exhibent, quoniam ne inter se quidem nisi paucissimis casibus exceptis comparari possunt. Ad innumerabilia alia autem quantitatum transcendentium genera calculus integralis perducit, dum omnibus formulis integralibus, quarum integratio algebraice expediri nequit, certae quantitates transcendentes designantur, in quarum natura evolvenda industria et sagacitas analystarum maxime cernitur. Cum igitur nunc quidem sit compertum omnes huiusmodi formulas integrales $\int V dx$, si V fuerit functio rationalis ipsius x, semper per logarithmos et arcus circulares exprimi posse, nisi forte algebraicam integrationem admittant, artificia integrandi pro iis casibus, quibus V est functio irrationalis ipsius x adhuc potissimum desiderantur, ubi quidem id imprimis esset optandum, ut eae formulae, quibus V est quantitas irrationalis, accuratius evolverentur, quarum integratio per arcus sive ellipticos sive hyperbolicos expediri queat. Atque in hac investigatione Auctor istius dissertationis imprimis est occupatus summumque studium contulit ad hanc formulam integralem $\int dx \sqrt{\frac{f+gxx}{h+kxx}}$ explicandam atque adeo ad arcus sive ellipticos sive hyperbolicos reducendam; quod negotium multo difficilius est, quam initio videatur. Prout enim quantitatum constantium f, g, h et k aliae fuerint vel positivae vel negativae, casus oriuntur natura sua maxime inter se discrepantes. Primo enim relatio inter has quatuor quantitates ita potest esse comparata, ut formula integralis arcum quendam sive ellipticum sive hyperbolicum simpliciter exprimat. Deinde fieri potest, ut integrale binis constet partibus, altera algebraica, altera arcum sive ellipticum sive hyperbolicum exprimente. Praeterea vero etiam eiusmodi dantur casus, quibus integrale neutro modo exhiberi potest, sed praeter partem algebraicam duos arcus, alterum ellipticum, alterum hyperbolicum requirit. In tractatione igitur formulae $\int dx \sqrt{\frac{f+gxx}{h+kxx}}$ ob istam varietatem Auctor coactus est duodecim casus con-

stituere, quos singulos operoso calculo ita feliciter expedivit, ut iam facile sit, quaecunque quantitates litteris f, g, h, k designentur, integrale concessa ellipsium et hyperbolarum rectificatione assignare. Saepenumero autem evenire potest, ut formulae integrales multo magis complicatae ope substitutionum idonearum ad talem formam perduci queant, quibus ergo omnibus casibus integratio expedita est censenda; ex quo haec investigatio calculo integrali haud leve incrementum attulisse est aestimanda.

Egregia omnino sunt, quae acutissimi Geometrae MACLAURIN[1]) et D'ALEMBERT[2]) de reductione formularum integralium ad rectificationem Ellipsis et Hyperbolae sunt commentati, cum in iis non solum insignis vis ingenii spectetur, sed etiam haud exigua spes affulgeat his rectificationibus in calculo aeque commode utendi, atque adhuc arcus circulares et logarithmos adhibere sumus soliti. Nullum enim est dubium, quin haec investigatio a summis Geometris tam felici successu suscepta latissime pateat atque uberrimos fructus aliquando sit allatura; quamvis enim iam plurimum in hoc negotio sit praestitum, minime tamen totum argumentum quasi exhaustum est censendum. Nam postquam longe diversa methodo usus eo perveni, ut tam in Ellipsi quam Hyperbola diversos arcus definire potuerim, quarum differentiam geometrice assignare liceat, de quo quidem laudati viri dubitasse videntur, hinc non levis accessio in tractatione huius argumenti expectari poterit. Imprimis autem hic idoneus signandi modus desiderari videtur, cuius ope arcus elliptici aeque commode in calculo exprimi queant, ac iam logarithmi et arcus circulares ad insigne Analyseos incrementum per idonea signa in calculum sunt introducti. Talia signa novam quandam calculi speciem suppeditabunt, cuius hic quasi prima elementa exponere constitui.

Quemadmodum autem omnes arcus circulares ad circulum, cuius radius unitati aequalis statuitur, referri solent, ita etiam pro omnibus sectionibus conicis, quas in calculum recipere volumus, mensuram quandam fixam unitate exprimendam assumi conveniet, quae ad omnes species aeque pertineat. Perspicuum autem est hanc mensuram axi transverso tribui non posse, cum is in parabola necessario fiat infinitus, in hyperbolis autem negativum valorem consequatur; aeque parum axis coniugatus ad hoc institutum est accommodatus, quippe qui in parabola quoque fit infinitus et in hyperbolis valorem adeo

1) C. MACLAURIN, *A Treatise of fluxions.* Edinburgh 1742, Vol. 2, p. 652. A. K.
2) Vide notam 1 p. 236. A. K.

imaginarium adipiscitur. Relinquitur igitur parameter, cui quominus perpetuo valor fixus tribui queat, nihil plane obstat; et quoniam pro circulo parameter abit in diametrum huiusque semissis unitate exprimi solet, constanter in sequentibus parametrum binario indicabo, ut eius semissis unitate exprimatur.

HYPOTHESIS 1

1. *Perpetuo igitur mihi unitas semiparametrum seu semilatus rectum sectionis conicae exprimat.*

COROLLARIUM 1

2. Si ergo a denotet semiaxem transversum, in quo abscissae x a vertice capiantur iisque applicatae y normaliter constituantur, habebitur ista aequatio

$$yy = 2x - \frac{xx}{a}.$$

COROLLARIUM 2

3. Quamdiu a quantitatem positivam denotat, aequatio erit pro ellipsi, quae quidem, si $a = 1$, abit in circulum; at posito $a = \infty$ habebitur parabola. Valores autem negativi ipsius a ad hyperbolas pertinent.

COROLLARIUM 3

4. Ex hac aequatione fit

$$dy = \frac{dx(a-x)}{\sqrt{a(2ax-xx)}}$$

hincque arcus abscissae x respondens

$$= \int \frac{dx\sqrt{(aa - 2a(1-a)x + (1-a)xx)}}{\sqrt{a(2ax-xx)}} \quad \text{seu} \quad = \int dx \sqrt{\left(\frac{a}{2ax-xx} + \frac{a-1}{a}\right)}$$

pro ellipsi, si fuerit a numerus positivus.

COROLLARIUM 4

5. Posito $a = 1$ fit pro circulo arcus abscissae x, quae est eius sinus versus, respondens $= \int dx \sqrt{\frac{1}{2x-xx}}$, uti constat, ac posito $a = \infty$ prodit parabolae arcus abscissae x respondens $= \int dx \sqrt{\left(\frac{1}{2x} + 1\right)}$.

COROLLARIUM 5

6. Si denique a habeat valorem negativum, puta $a = -c$, erit pro hyperbolis arcus abscissae x respondens $= \int dx \sqrt{\left(\frac{c}{2cx + xx} + \frac{c+1}{c}\right)}$.

HYPOTHESIS 2

7. *In sectione conica, cuius semiparameter* $= 1$ *et semiaxis transversus* $= a$ *atque abscissae in axe transverso a vertice capiantur, arcum abscissae* x *respondentem hac scriptione* $\Pi x[a]$ *indicabo.*

COROLLARIUM 1

8. Post signum ergo Π scribetur abscissa in axe transverso a vertice computata, cui subiungetur semiaxis transversus intra uncinulas [ ] expressus.

COROLLARIUM 2

9. Haec ergo expressio $\Pi x[a]$ designat arcum ellipticum, si a sit quantitas positiva, et circularem quidem, si $a = 1$, cuius sinus versus $= x$. At si $a = \infty$, exprimit ea arcum parabolicum, ac denique si a sit quantitas negativa, arcum hyperbolicum.

COROLLARIUM 3

10. Habet ergo huiusmodi expressio $\Pi x[a]$ valorem determinatum eaque non solum sectio conica definitur, sed etiam eius arcus illa expressione indicatur.

COROLLARIUM 4

11. Manifestum autem est, ut istius expressionis valor fiat realis, abscissam x non solum realem, sed etiam positivam esse debere. Tum vero praeterea, si fuerit a quantitas positiva, necesse est, ut abscissa x limitem $2a$ non transgrediatur. Quantitatem a autem necessario realem esse oportet.

COROLLARIUM 5

12. Haec ergo expressio $\Pi x[a]$ imaginaria erit, si vel 1^{mo} numerus a fuerit imaginarius, vel 2^{do} x quantitas imaginaria, vel 3^{tio} quantitas negativa, vel 4^{to} positiva quidem, sed maior quam $2a$, si scilicet a sit quantitas positiva.

COROLLARIUM 6

13. Notetur quoque hanc formulam $\Pi x[a]$ eiusmodi functionem ipsius x exhibere, quae evanescat evanescente x, ita ut sit $\Pi 0[a] = 0$. Sin autem sit x quantitas infinite parva $= \omega$, erit $\Pi\omega[a] = \sqrt{2\omega}$ neque ergo ab a pendet.

THEOREMA 1

14. *Si haec formula differentialis* $dx\sqrt{\left(\frac{a}{2ax-xx}+\frac{a-1}{a}\right)}$ *ita integretur, ut integrale evanescat posito* $x = 0$, *erit*

$$\int dx\sqrt{\left(\frac{a}{2ax-xx}+\frac{a-1}{a}\right)} = \Pi x[a].$$

DEMONSTRATIO

Utraque enim expressio refertur ad sectionem conicam, cuius semiparameter $= 1$ et semiaxis transversus $= a$, atque arcum eius denotat a vertice sumtum, qui abscissae x respondet abscissa in axe transverso sumta ac pariter a vertice computata.

COROLLARIUM 1

15. Si pro a scribamus $-a$, habebitur

$$\int dx\sqrt{\left(\frac{a}{2ax+xx}+\frac{a+1}{a}\right)} = \Pi x[-a],$$

quo casu, si quantitas uncinulis inclusa sit negativa, arcus hyperbolicus indicatur.

COROLLARIUM 2

16. Si sit $a = \infty$, quo casu prodit rectificatio parabolae, erit

$$\int dx\sqrt{\left(\frac{1}{2x}+1\right)} = \Pi x[\infty],$$

cuius valor, uti constat, per logarithmos exhiberi potest.

COROLLARIUM 3

17. At si sit $a = 1$, ut habeatur

$$\int \frac{dx}{\sqrt{(2x - xx)}} = \Pi x[1],$$

hac expressione arcus circuli, cuius radius $= 1$, exprimitur, cuius sinus versus $= x$; eius ergo cosinus erit $= 1 - x$ et sinus $= \sqrt{(2x - xx)}$.

COROLLARIUM 4

18. Cum eidem abscissae x geminus arcus, alter positivus, alter negativus, respondeat, expressio $\Pi x[a]$ per se geminum exhibebit valorem, perinde uti signa radicalia quadratica; erit ergo functio biformis, tam valorem negativum quam positivum continens.

SCHOLION

19. Quoties autem expressio $\Pi x[a]$ ad ellipsin refertur, ea non solum duos, verum adeo infinitos valores complectitur, perinde uti in circulo infiniti dantur arcus eidem sinui verso x convenientes. Naturam ergo huius functionis infinitiformis pro ellipsibus accuratius perpendamus.

PROBLEMA 1

20. *Invenire omnes arcus ellipticos eidem abscissae x respondentes seu definire omnes valores formulae $\Pi x[a]$ convenientes.*

SOLUTIO

Sit z minimus arcus abscissae x respondens in ellipsi, cuius semiaxis transversus est $= a$; ponatur semiperimeter ellipsis $= A$, ut sit tota perimeter $= 2A$, atque manifestum est eidem abscissae x etiam respondere arcus $2A - z$, $2A + z$, $4A - z$, $4A + z$, $6A - z$, $6A + z$ etc., qui omnes cum suis negativis continentur in formula $\Pi x[a]$, ita ut eius valor in genere sit $\pm 2nA \pm z$ denotante n numerum integrum quemcunque.

COROLLARIUM 1

21. Cum $\frac{1}{2}A$ sit quarta pars perimetri ellipsis eique abscissa $x = a$ conveniat, erit $\frac{1}{2}A = \Pi a\,[a]$, semiperimetro autem A convenit abscissa $2a$, unde $A = \Pi 2a\,[a]$, ergo $\Pi 2a\,[a] = 2\Pi a\,[a]$.

COROLLARIUM 2

22. Si capiatur abscissa $= 2a - x$, erit arcus ei respondens $= A - \Pi x\,[a]$, unde colligitur haec aequalitas

$$\Pi x\,[a] + \Pi(2a - x)\,[a] = 2\Pi a\,[a],$$

ubi vincula ( ), quibus abscissa inscribitur, ab uncinulis [ ] semiaxem transversum continentibus probe distingui oportet.

COROLLARIUM 3

23. Eadem aequalitas ex integrali potest colligi; posito enim $2a - x$ loco x, erit

$$\Pi(2a - x)\,[a] = -\int dx \sqrt{\left(\frac{a}{2ax - xx} + \frac{a-1}{a}\right)} = -\,\Pi x\,[a] + \text{Const.}$$

Constans vero ex quodam casu debet colligi. Scilicet si ponatur $x = 0$, fit Const. $= \Pi 2a\,[a]$; vel si ponatur $x = a$, prodit

$$\text{Const.} = \Pi a\,[a] + \Pi a\,[a] = 2\Pi a\,[a].$$

SCHOLION

24. Arcus elliptici praeterea hanc habent proprietatem, ut, si axis transversus $2a$ minor fuerit parametro, quod scilicet evenit, si axis minor pro transverso capiatur, iidem arcus sumi possint in alia ellipsi, cuius axis sit maior parametro. Nititur haec reductio similitudine ellipsium, quarum semiaxes sunt a et $\frac{1}{a}$ manente parametro eadem $= 2$.

PROBLEMA 2

25. *Arcum ellipticum $\Pi x[a]$, si fuerit $a<1$, ad aliam ellipsin reducere, cuius semiaxis sit unitate maior.*

SOLUTIO

Cum sit

$$\Pi x[a] = \int dx \sqrt{\left(\frac{a}{2ax-xx}+1-\frac{1}{a}\right)},$$

statuatur

$$\sqrt{(2ax-xx)} = a - aay$$

eritque

$$2ax - xx = aa(1-2ay+aayy) \quad \text{hincque} \quad a - x = a\sqrt{(2ay-aayy)};$$

unde fit

$$dx = \frac{-aady(1-ay)}{\sqrt{(2ay-aayy)}}.$$

Facta hac substitutione consequemur

$$\Pi x[a] = \int \frac{-aady(1-ay)}{\sqrt{(2ay-aayy)}} \sqrt{\left(\frac{1}{a(1-ay)^2}+1-\frac{1}{a}\right)}$$

seu

$$\Pi x[a] = -a\sqrt{a}\cdot\int dy \sqrt{\frac{1+(a-1)(1-ay)^2}{2ay-aayy}},$$

quae expressio reducitur ad hanc formam

$$\Pi x[a] = -a\sqrt{a}\cdot\int dy \sqrt{\left(\frac{1}{2y-ayy}+1-a\right)}.$$

Ponamus in formula integrali $a=\frac{1}{b}$, ut sit $b=\frac{1}{a}$, ac fiet ea

$$\int dy \sqrt{\left(\frac{b}{2by-yy}+1-\frac{1}{b}\right)} = \Pi y[b] + \text{Const.}$$

Quare restituta littera a obtinebitur ob $y=\frac{a-\sqrt{(2ax-xx)}}{aa}$

$$\Pi x[a] = \text{Const.} - a\sqrt{a}\cdot \Pi\frac{a-\sqrt{(2ax-xx)}}{aa}\left[\frac{1}{a}\right],$$

ubi ex casu $x=0$ definitur constans $= a\sqrt{a}\cdot\Pi\frac{1}{a}\left[\frac{1}{a}\right]$, ita ut sit

$$\Pi x[a] = a\sqrt{a}\cdot\Pi\frac{1}{a}\left[\frac{1}{a}\right] - a\sqrt{a}\cdot\Pi\frac{a-\sqrt{(2ax-xx)}}{aa}\left[\frac{1}{a}\right],$$

sicque arcus ellipsis, cuius semiaxis est a, reductus est ad arcus alius ellipsis, cuius semiaxis est $=\frac{1}{a}$.

COROLLARIUM 1

26. Si ponatur $x=a$, fit

$$\frac{a-\sqrt{(2ax-xx)}}{aa}=0 \quad \text{ideoque} \quad \Pi a[a] = a\sqrt{a}\cdot\Pi\frac{1}{a}\left[\frac{1}{a}\right].$$

Scilicet perimeter prioris ellipsis, cuius semiaxis $=a$, est ad perimetrum posterioris, cuius semiaxis $=\frac{1}{a}$, uti $a\sqrt{a}$ ad 1 seu ut $a^{\frac{3}{4}}$ ad $\frac{1}{a^{\frac{3}{4}}}$.

COROLLARIUM 2

27. Cum arcus abscissae $\frac{a-\sqrt{(2ax-xx)}}{aa}$ respondens posito $x=0$ fiat $=\Pi\frac{1}{a}\left[\frac{1}{a}\right]$, hinc aucto x decrescat, donec evanescat posito $x=a$, lex continuitatis exigit, ut sumto $x>a$ iste arcus negativum obtineat valorem.

COROLLARIUM 3

28. Facto ergo $x=2a$ erit

$$\Pi\frac{a-\sqrt{(2ax-xx)}}{aa}\left[\frac{1}{a}\right] = -\Pi\frac{1}{a}\left[\frac{1}{a}\right],$$

quo notato fiet hoc casu $x=2a$

$$\Pi 2a[a] = 2\Pi a[a] = 2a\sqrt{a}\,\Pi\frac{1}{a}\left[\frac{1}{a}\right],$$

id quod consentit cum coroll. 1.

SCHOLION

29. Substitutione hic adhibita $\sqrt{(2ax-xx)} = a - aay$ formulam integralem in aliam sui similem transmutavimus, cuius valor per arcum alius ellipsis exhiberi poterat. Si autem aliis substitutionibus utamur, semper

adipiscimur formulas integrales, quarum integratio per rectificationem sectionum conicarum expediri potest; quia vero a tam negativum quam positivum valorem recipere potest, substitutiones eaedem tam ad ellipses quam hyperbolas extendi possunt.

PROBLEMA 3

30. *Formulam integralem*

$$\int dx\sqrt{\left(\frac{a}{2ax-xx}+1-\frac{1}{a}\right)}$$

per substitutiones idoneas in alias formulas concinniores transformare, quarum valor semper futurus sit $=\Pi x[a]$.

SOLUTIO

Prima reductio fit ponendo $x=a-naz$, quo facto formula integralis induit hanc formam

$$\int -ndz\sqrt{\frac{aa-nna(a-1)zz}{1-nnzz}}=\Pi a(1-nz)[a];$$

multiplicetur ea per m, ut sit

$$\int -dz\sqrt{\frac{m^2n^2aa+m^2n^4a(a-1)zz}{1-nnzz}}=m\Pi a(1-nz)[a],$$

quam expressionem iam ad hanc formam, concinnam aeque ac generalem, reducere licet

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}};$$

fieri scilicet oportet

$$m^2n^2a^2h=f,\quad m^2n^4a(1-a)h=g,\quad -nnh=k,$$

unde ob $nnh=-k$ et $n=\sqrt{\frac{-k}{h}}$ erit

$$-mmaak=f,\quad mma(1-a)kk=gh$$

hincque

$$\frac{(a-1)k}{a}=\frac{gh}{f}\quad\text{et}\quad a=\frac{fk}{fk-gh}.$$

Porro est

$$m = \frac{1}{a}\sqrt{\frac{-f}{k}} \quad \text{seu} \quad m = \frac{fk-gh}{fk}\sqrt{\frac{-f}{k}},$$

ex quibus valoribus concluditur fore

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = C - \frac{fk-gh}{fk}\sqrt{\frac{-f}{k}}\, \Pi \frac{fk}{fk-gh}\left(1 - z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-gh}\right].$$

Hoc ergo integrale, nisi forma sit imaginaria, per rectificationem ellipsis absolvitur, si fuerit $\frac{fk}{fk-gh}$ quantitas positiva; sin autem sit negativa, integratio arcum hyperbolicum indicat.

COROLLARIUM 1

31. Ut ergo haec forma ab imaginariis sit libera, necesse est, ut tam $\frac{-f}{k}$ quam $\frac{-k}{h}$ sit quantitas positiva. Si alterutra vel ambae fuerint negativae, expressio imaginariis implicatur; nihilo vero minus eius valor erit realis, si modo differentiale ipsum sit reale.

COROLLARIUM 2

32. Cum autem formula differentialis ponatur realis, assumere licet tam $f+gzz$ quam $h+kzz$ esse quantitates positivas; si enim ambae essent negativae, mutatis signis ad positivas reduci possent. Ita statuamus esse

$$f+gzz > 0 \quad \text{et} \quad h+kzz > 0.$$

COROLLARIUM 3

33. Ut autem formula nostra inventa arcum realem sectionis conicae exprimat, non sufficit esse $\sqrt{\frac{-f}{k}}$ et $\sqrt{\frac{-k}{h}}$ quantitates reales, sed praeterea requiritur, ut abscissa sit positiva; ubi duos casus perpendi convenit, prout sectio conica fuerit ellipsis vel hyperbola.

COROLLARIUM 4

34. Sit ergo primo sectio conica ellipsis seu $\frac{fk}{fk-gh}$ quantitas positiva atque necesse est, ut sit $1 - z\sqrt{\frac{-k}{h}} > 0$ seu $1 > \frac{-kzz}{h}$, unde fit $\frac{h+kzz}{h} > 0$. At per hypothesin est $h+kzz > 0$. Quare casu, quo $\frac{fk}{fk-gh} > 0$, ad realitatem insuper requiritur, ut h sit quantitas positiva.

COROLLARIUM 5

35. Pro hyperbola, seu si $\frac{fk}{fk-gh}$ fuerit quantitas negativa, integratio nostra ita debet repraesentari

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{gh-fk}{fk}\sqrt{\frac{-f}{k}}\,\Pi\frac{fk}{gh-fk}\left(z\sqrt{\frac{-k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right],$$

ita ut $\frac{fk}{gh-fk}$ iam sit quantitas positiva. Necesse autem est, ut sit

$$z\sqrt{\frac{-k}{h}} > 1 \quad \text{seu} \quad \frac{h+kzz}{h} < 0,$$

quare ob $h+kzz>0$ arcus hyperbolicus non erit realis, nisi sit h quantitas negativa.

COROLLARIUM 6

36. Pro ellipsi ergo, seu si sit $\frac{fk}{fk-gh}>0$, nostra expressio arcum continebit realem, si fuerit

$$1.\ h>0, \quad 2.\ k<0 \quad \text{ac} \quad 3.\ f>0.$$

Pro hyperbola autem, seu si $\frac{fk}{gh-fk}>0$, arcus erit realis, si fuerit

$$1.\ h<0, \quad 2.\ k>0 \quad \text{et} \quad 3.\ f<0.$$

SCHOLION 1

37. Ope formulae igitur inventae nonnisi aliquot casus integralis propositi

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

expedire possumus. Nempe cum in genere litterae f, g, h et k quantitates sive positivas sive negativas significent, si iam ad casus descendentes eas tantum pro positivis assumamus, sequentes integrationes reales consequemur

$$\text{I.}\ \int dz\sqrt{\frac{f+gzz}{h-kzz}} = C - \frac{fk+gh}{fk}\sqrt{\frac{f}{k}}\,\Pi\frac{fk}{fk+gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk+gh}\right].$$

$$\text{II.}\ \int dz\sqrt{\frac{f-gzz}{h-kzz}} = C - \frac{fk-gh}{fk}\sqrt{\frac{f}{k}}\,\Pi\frac{fk}{fk-gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk-gh}\right];$$

at hoc casu requiritur, ut sit $fk>gh$.

$$\text{III.}\quad \int dz \sqrt{\frac{-f+gzz}{-h+kzz}} = C + \frac{gh-fk}{fk}\sqrt{\frac{f}{k}}\,\Pi\frac{fk}{gh-fk}\left(z\sqrt{\frac{k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right];$$

hoc vero casu requiritur, ut sit $gh > fk$.

Cum igitur in hoc tertio casu indoles litterarum f, h, k iam sit definita, pro g autem quantitatem negativam assumere non liceat, hos tantum tres casus per nostrum problema expedire licet. Reliqui vero omnes excluduntur, dum ad arcus imaginarios perducuntur. Interim tamen cum certo habeant valores reales, quemadmodum hi per alios arcus reales exprimi queant, in sequentibus investigabimus.

SCHOLION 2

38. Antequam autem hoc opus suscipiamus, e re erit omnes casus pro diversitate signorum, quibus litterae f, g, h, k affectae esse possunt, enumerare, ubi etiam fieri potest, ut quidam ob aliam conditionem in binos subdividi debeant, quemadmodum supra in secundo et tertio usu venit. Hac conditione adiecta sequentes 12 habebimus casus, ubi quidem litterae f, g, h, k tantum positivos valores habere accipiuntur.

I. $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$, si fuerit $fk > gh$.

II. $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$, si fuerit $gh > fk$.

III. $\int dz \sqrt{\frac{f+gzz}{h-kzz}}$ nulla limitatione adiuncta.

IV. $\int dz \sqrt{\frac{f+gzz}{kzz-h}}$ nulla limitatione adiuncta.

V. $\int dz \sqrt{\frac{f-gzz}{h+kzz}}$ nulla limitatione adiuncta.

VI. $\int dz \sqrt{\frac{f-gzz}{h-kzz}}$, si fuerit $fk > gh$.

VII. $\int dz \sqrt{\frac{f-gzz}{h-kzz}}$, si fuerit $fk < gh$.

VIII. $\int dz \sqrt{\frac{f-gzz}{-h+kzz}}$; hic necessario est $fk > gh$.

IX. $\int dz \sqrt{\frac{-f+gzz}{h+kzz}}$ nulla limitatione adiuncta.

$$\text{X. } \int dz\sqrt{\frac{-f+gzz}{h-kzz}}; \text{ hic necessario est } fk<gh.$$

$$\text{XI. } \int dz\sqrt{\frac{-f+gzz}{-h+kzz}}, \text{ si fuerit } fk>gh.$$

$$\text{XII. } \int dz\sqrt{\frac{-f+gzz}{-h+kzz}}, \text{ si fuerit } fk<gh.$$

Atque ex his duodecim casibus hactenus tantum tres, scilicet III, VI ac XII, conficere licuit, quorum integralia per arcus simplices sectionum conicarum exprimuntur.

SCHOLION 3

39. Quanquam autem his tribus casibus integralia per arcus sive ellipticos sive hyperbolicos expressimus, tamen quaedam dantur relationes inter litteras f, g, h et k, quibus nostra expressio tantis incommodis implicatur, ut verus valor integralis inde erui nequeat, etiamsi per se sit perquam facilis. Ac primo quidem in genere, si in formula

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

fuerit $fk=gh$, valor integralis ita quantitatibus evanescentibus et infinitis involvitur, ut eius vera quantitas inde perspici nequeat, cum tamen ea per se sit planissima; posito enim $k=\frac{gh}{f}$ erit

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}=\int dz\sqrt{\frac{f(f+gzz)}{h(f+gzz)}}=\int dz\sqrt{\frac{f}{h}}=C+z\sqrt{\frac{f}{h}},$$

ita ut revera sit ob $gh=fk$

$$C-\frac{fk-fk}{fk}\sqrt{\frac{-f}{k}}\,\varPi\frac{fk}{fk-fk}\left(1-z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-fk}\right]=z\sqrt{\frac{f}{h}},$$

etsi ratio huius aequalitatis difficulter perspiciatur, cum haec formula potius arcum parabolicum abscissae infinitae respondentem, qui autem per factorem evanescentem sit multiplicatus, indicare videatur. Interim tamen si perpendamus in parabola arcum, qui abscissae infinitae respondeat, ad abscissam rationem aequalitatis habere, erit

$$\varPi\frac{fk}{fk-fk}\left(1-z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-fk}\right]=\frac{fk}{fk-fk}\left(1-z\sqrt{\frac{-k}{h}}\right),$$

qui arcus per factorem

$$\frac{-(fk-fk)}{fk}\sqrt{\frac{-f}{k}}$$

multiplicatus praebet productum finitum

$$=-\left(1-z\sqrt{\frac{-k}{h}}\right)\sqrt{\frac{-f}{k}}=-\sqrt{\frac{-f}{k}}+z\sqrt{\frac{f}{h}},$$

qui valor cum veritate egregie congruit. Reliquas difficultates casus particulares percurrentes seorsim examinemus.

INTEGRATIO CASUS III

$$\int dz\sqrt{\frac{f+gzz}{h-kzz}}=C-\frac{fk+gh}{fk}\sqrt{\frac{f}{k}}\,\Pi\frac{fk}{fk+gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk+gh}\right]$$

40. Si f, g, h, k denotent quantitates nihilo maiores, arcus ellipticus pro integrali facile assignatur; neque turbat casus, quo $g=0$, quippe qui per arcum circularem expeditur eritque

$$\int\frac{dz\sqrt{f}}{\sqrt{(h-kzz)}}=C-\sqrt{\frac{f}{k}}\,\Pi\left(1-z\sqrt{\frac{k}{h}}\right)[1].$$

Deinde h evanescere nequit, quin simul formula differentialis ipsa fiat imaginaria. At si f vel k evanescat, quorum priori casu integrale est algebraicum, posteriori vero per logarithmos dari potest, nostra formula refertur ad ellipsin evanescentem nihilque inde concludere licet; mox autem pro eodem casu aliam integralis formam exhibebimus, unde vera integralis quantitas facilius elici poterit.

INTEGRATIO CASUS VI

$$\int dz\sqrt{\frac{f-gzz}{h-kzz}}=C-\frac{fk-gh}{fk}\sqrt{\frac{f}{k}}\,\Pi\frac{fk}{fk-gh}\left(1-z\sqrt{\frac{k}{h}}\right)\left[\frac{fk}{fk-gh}\right]$$

SI FUERIT $fk>gh$

41. Hic iterum nulla difficultas occurrit, quicunque valores litteris f, g, h et k tribuantur, dummodo sit $fk>gh$; semper enim integrale per arcum ellipticum exprimitur neque etiam negotium facessit casus $g=0$, quo ut ante arcus circularis denotatur. At si sit $k=0$, neque enim f et h in nihi-

lum abire possunt, conditio $fk > gh$ non amplius salvari potest sicque hic nullum incommodum locum habet, praeter id, quo est $fk = gh$, quod autem iam ante in genere expedivimus.

INTEGRATIO CASUS XII

$$\int dz\sqrt{\frac{-f+gzz}{-h+kzz}} = C + \frac{gh-fk}{fk}\sqrt{\frac{f}{k}}\,\varPi_{\frac{fk}{gh-fk}}\left(z\sqrt{\frac{k}{h}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

SI FUERIT $gh > fk$

42. Hoc casu integrale arcu hyperbolico definitur, ubi neque g neque k potest fieri negativum. Si fuerit $f = 0$, quo casu integrale fit algebraicum, axis hyperbolae evanescit neque hinc valor integralis cognoscitur. At si $h = 0$, conditio necessaria $gh > fk$ evertitur, difficultas ergo tantum casu $f = 0$ subsistit, quae autem in aliis formulis infra pro eodem hoc casu tradendis tolletur.

PROBLEMA 4

43. *Formulam integralem*

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

per substitutionem in aliam sui similem transformare.

SOLUTIO

Tentanti huiusmodi substitutionem $z = \sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$ patebit sumi debere $x = \sqrt{(h+kzz)}$; unde fit

$$z = \sqrt{\frac{xx-h}{k}}, \quad dz = \frac{xdx}{\sqrt{k(xx-h)}} \quad \text{et} \quad f+gzz = \frac{fk-gh+gxx}{k}$$

ideoque

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{1}{k}\int dx\sqrt{\frac{fk-gh+gxx}{xx-h}},$$

quae locum habet, quoties k est quantitas positiva, quoniam tum $xx-h = kzz$ est quantitas positiva. Sin autem k fuerit quantitas negativa ideoque $h-xx$ positiva, transformatio ita est repraesentanda

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \frac{1}{k}\int dx\sqrt{\frac{gh-fk-gxx}{h-xx}}.$$

COROLLARIUM 1

44. Comparantes formulam

$$\int dx \sqrt{\frac{fk-gh+gxx}{xx-h}}$$

cum formula initio generatim integrata

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}}$$

habebimus $z=x$, $f=fk-gh$, $g=g$, $h=-h$, $k=1$, unde $fk-gh=fk$ et integrale

$$\int dx \sqrt{\frac{fk-gh+gxx}{xx-h}} = C - \frac{fk}{fk-gh} \sqrt{(-fk+gh)}\, \varPi \frac{fk-gh}{fk}\left(1-x\sqrt{\frac{1}{h}}\right)\left[\frac{fk-gh}{fk}\right].$$

COROLLARIUM 2

45. Substituto hoc valore, cum sit $x=\sqrt{(h+kzz)}$, erit

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{f}{\sqrt{(gh-fk)}}\, \varPi \frac{gh-fk}{fk}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-gh+fk}{fk}\right],$$

qui arcus ut sit realis, necesse est, ut primo sit $gh-fk>0$, tum vero etiam $h>0$. Erit ergo ad hyperbolam, si fk sit quantitas positiva et k positiva; ad ellipsin esse nequit, nisi sit k quantitas negativa, f vero positiva, quia hoc casu esse debet fk quantitas negativa.

COROLLARIUM 3

46. Simili modo altera forma $\int dx \sqrt{\frac{gh-fk-gxx}{h-xx}}$ cum canonica $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ comparata dat $z=x$, $f=gh-fk$, $g=-g$, $h=h$ et $k=-1$; unde $fk-gh=fk$ et

$$\int dx \sqrt{\frac{gh-fk-gxx}{h-xx}} = C - \frac{fk}{fk-gh} \sqrt{(gh-fk)}\, \varPi \frac{fk-gh}{fk}\left(1-x\sqrt{\frac{1}{h}}\right)\left[\frac{fk-gh}{fk}\right].$$

COROLLARIUM 4

47. Substituto ergo pro x valore $\sqrt{(h+kzz)}$ erit ut ante

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{f}{\sqrt{(gh-fk)}}\, \varPi \frac{gh-fk}{-fk}\left(1-\frac{\sqrt{(h+kzz)}}{\sqrt{h}}\right)\left[\frac{gh-fk}{-fk}\right],$$

quae locum habere nequit, nisi $gh - fk$ et h sit quantitas positiva. Pro ellipsi erit, si k sit quantitas negativa et f positiva, contra autem pro hyperbola, si k et g sint positivae, quemadmodum iam ante definivimus, ita ut hos duos casus distinguere non opus fuerit.

COROLLARIUM 5

48. Geminis his integralibus formulae generalis $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ inter se collatis habebimus

$$\varPi \frac{fk}{fk-gh}\left(1 - z\sqrt{\frac{-k}{h}}\right)\left[\frac{fk}{fk-gh}\right] = \frac{-fk\sqrt{fk}}{(fk-gh)^{\frac{3}{2}}}\varPi \frac{fk-gh}{fk}\left(1 - \frac{\sqrt{(h+kzz)}}{\sqrt{h}}\right)\left[\frac{fk-gh}{fk}\right],$$

quae aequalitas posito ad abbreviandum

$$\frac{fk}{fk-gh} = \frac{m}{n} \quad \text{et} \quad z\sqrt{\frac{-k}{h}} = t$$

abit in hanc formam

$$\varPi \frac{m}{n}(1-t)\left[\frac{m}{n}\right] = \frac{-m\sqrt{m}}{n\sqrt{n}}\varPi \frac{n}{m}\left(1 - \sqrt{(1-tt)}\right)\left[\frac{n}{m}\right].$$

COROLLARIUM 6

49. Arcus igitur ellipticus quicunque respondens abscissae $= 1 - t$ semiaxe existente $= \frac{m}{n}$ reducitur ad arcum alius ellipsis, cuius semiaxis est $= \frac{n}{m}$ et abscissa $= 1 - \sqrt{(1-tt)}$, hunc arcum per $\frac{m\sqrt{m}}{n\sqrt{n}}$ multiplicando, cuius aequalitatis ratio est similitudo harum duarum ellipsium. Simili autem modo arcus hyperbolicus ad alium reduci nequit, quia ob $\frac{m}{n}$ negativum fit $\sqrt{\frac{m}{n}}$ imaginarium.

SCHOLION

50. Hinc novas integrationes nanciscimur realiter expressas; primam suggerit § 45 arcum hyperbolicum involventem, ubi hae conditiones requiruntur

$$1.\ h > 0, \quad 2.\ k > 0, \quad 3.\ f > 0 \quad \text{et} \quad 4.\ gh > fk,$$

unde ob $h > 0$ erit quoque $g > 0$; hisque casus II § 38 enumeratorum continetur. Deinde arcus ellipticus negotium conficiet his conditionibus

$$1.\ k < 0, \quad 2.\ h > 0, \quad 3.\ gh - fk > 0 \quad \text{et} \quad 4.\ f > 0 \quad \text{ob} \quad -fk > 0,$$

unde g etiam nunc positive et negative capi potest. Si sumatur positive, prodit casus III, sin negative, casus VI, qui quidem iam supra sunt soluti. Verum in genere notandum omnes arcus ellipticos duplici modo exprimi posse per paragraphum praecedentem. Integralia ergo horum trium casuum ita se habebunt.

INTEGRATIO CASUS II

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = C + \frac{f}{\sqrt{(gh-fk)}}\, \Pi \frac{gh-fk}{fk}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

SI FUERIT $gh > fk$

51. Ob conditionem $gh > fk$ neque g neque h evanescere potest. Si f evanescat, hyperbola abit in parabolam, cuius arcus abscissae infinitae respondens hic indicatur, qui ergo abscissae aequalis est censendus; unde pro casu $f=0$ habebitur istud integrale

$$\int dz \sqrt{\frac{gzz}{h+kzz}} = C + \frac{\sqrt{gh}}{k}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right) = C + \frac{\sqrt{g(h+kzz)}}{k},$$

quod veritati omnino est consentaneum.

SCHOLION

52. Alter casus moram facessens est, quo $k=0$ et hyperbola iterum abit in parabolam. At ob $k=0$ erit

$$\sqrt{\left(1+\frac{kzz}{h}\right)}-1 = \frac{kzz}{2h};$$

unde integratio per arcum parabolicum absolvetur hoc modo

$$\int dz \sqrt{\frac{f+gzz}{h}} = C + \frac{f}{\sqrt{gh}}\, \Pi \frac{gzz}{2f}[\infty],$$

quae eadem operatione consueta elicitur. Si insuper esset $g=0$, ob

$$\Pi \frac{gzz}{2f} = z\sqrt{\frac{g}{f}}$$

(§ 13) foret

$$\int dz \sqrt{\frac{f}{h}} = C + z\sqrt{\frac{f}{h}}.$$

INTEGRATIO CASUS III

$$\int dz \sqrt{\frac{f+gzz}{h-kzz}} = C + \frac{f}{\sqrt{(gh+fk)}}\, \Pi\, \frac{gh+fk}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{gh+fk}{fk}\right]$$

SINE ULLA LIMITATIONE

53. Hinc casus $h=0$ sponte excluditur; unde tres singulares relinquuntur. Si primo sit $k=0$, ellipsis abit in parabolam et ob

$$1-\sqrt{\left(1-\frac{kzz}{h}\right)} = \frac{kzz}{2h}$$

habetur ut ante

$$\int dz \sqrt{\frac{f+gzz}{h}} = C + \frac{f}{\sqrt{gh}}\, \Pi\, \frac{gzz}{2f}[\infty];$$

si deinde sit $f=0$, denuo parabola et arcus abscissae infinitae respondens ideoque aequalis censendus prodit, unde fit

$$\int dz \sqrt{\frac{gzz}{h-kzz}} = C + \frac{\sqrt{gh}}{k}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right) = C - \frac{\sqrt{g(h-kzz)}}{k};$$

si tertio sit $g=0$, ellipsis abit in circulum fitque

$$\int dz \sqrt{\frac{f}{h-kzz}} = C + \frac{f}{\sqrt{fk}}\, \Pi \left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)[1];$$

sicque casus difficiliores supra § 40 memoratos hic expedivimus.

INTEGRATIO CASUS VI

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}} = C + \frac{f}{\sqrt{(fk-gh)}}\, \Pi\, \frac{fk-gh}{fk}\left(1-\frac{\sqrt{(h-kzz)}}{\sqrt{h}}\right)\left[\frac{fk-gh}{fk}\right]$$

SI MODO FUERIT $fk>gh$

54. Hoc casu aeque ac supra § 41, ubi idem est pertractatus, nulla difficultas relinquitur, quia ob $fk>gh$ neque f neque k evanescere potest neque vero etiam h in nihilum abire potest, quin denominator $h-kzz$ fiat negativus. At si $g=0$, nulla occurrit difficultas, cum integratio ad arcum circularem revocetur.

SCHOLION

55. Hac ergo reductione id sumus lucrati, ut iam praeter casus III, VI et XII ante evolutos etiam casum II expediverimus. Reliqui vero octo casus nullo modo per arcus simplices reales integrari possunt, sed praeterea partem algebraicam continent; quin etiam nonnulli praeter hanc partem algebraicam binos arcus, alterum ellipticum, alterum hyperbolicum, complectuntur. Ad haec autem integralia investiganda necesse est, ut alias formulas integrales affines praeter variabilem z bina radicalia $\sqrt{(f+gzz)}$ et $\sqrt{(h+kzz)}$ involventes contemplemur, quae ad formam $\int dx \sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$ sint reductibiles.

PROBLEMA 5

56. *Alias formulas integrales praeter z bina radicalia*

$$\sqrt{(f+gzz)} \quad et \quad \sqrt{(h+kzz)}$$

continentes invenire, quarum integratio ad formam

$$\int dx \sqrt{\frac{\alpha+\beta xx}{\gamma+\delta xx}}$$

reduci queat.

SOLUTIO

Fit hoc substitutionibus, quarum praecipuas hic percurramus.

1. Sit $x=\frac{1}{z}$; erit $z=\frac{1}{x}$, $\sqrt{(f+gzz)}=\frac{\sqrt{(fxx+g)}}{x}$ et $\sqrt{(h+kzz)}=\frac{\sqrt{(hxx+k)}}{x}$. Hinc ob $dx=-\frac{dz}{zz}$ erit

$$dx\sqrt{\frac{fxx+g}{hxx+k}}=-\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}$$

ideoque

$$\int\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}=-\int dx\sqrt{\frac{fxx+g}{hxx+k}}.$$

2. Ponatur $x=\sqrt{(f+gzz)}$; erit $dx=\frac{gzdz}{\sqrt{(f+gzz)}}$, $z=\sqrt{\frac{xx-f}{g}}$ et $\sqrt{(h+kzz)}=\sqrt{\frac{gh-fk+kxx}{g}}$, unde conficitur

$$dx\sqrt{\frac{xx-f}{gh-fk+kxx}}=\frac{gzzdz}{\sqrt{(f+gzz)(h+kzz)}}$$

et

$$dx\sqrt{\frac{gh-fk+kxx}{xx-f}}=gdz\sqrt{\frac{h+kzz}{f+gzz}}\quad\text{(omittenda)},$$

quare erit

$$\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{1}{g}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}}.$$

3. Si ponatur $x=\sqrt{(h+kzz)}$, erit simili modo

$$\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{1}{k}\int dx\sqrt{\frac{xx-h}{fk-gh+gxx}}.$$

4. Sit $x=\frac{1}{\sqrt{(f+gzz)}}$; erit $dx=\frac{-gzdz}{(f+gzz)^{\frac{3}{2}}}$, $z=\frac{\sqrt{(1-fxx)}}{x\sqrt{g}}$, $\sqrt{(f+gzz)}=\frac{1}{x}$ et $\sqrt{(h+kzz)}=\sqrt{\frac{k+(gh-fk)xx}{gxx}}$, unde fit

$$dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}}=\frac{-gzzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}$$

et

$$dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}}=\frac{-gdz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}$$

hincque

$$\int\frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=-\frac{1}{g}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}}$$

et

$$\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}=-\frac{1}{g}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}}.$$

5. Simili modo si ponatur $x=\frac{1}{\sqrt{(h+kzz)}}$, reperitur

$$\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=-\frac{1}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}},$$

$$\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}}=-\frac{1}{k}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}}.$$

6. Ponatur $x=\frac{z}{\sqrt{(f+gzz)}}$; erit $dx=\frac{fdz}{(f+gzz)^{\frac{3}{2}}}$, tum $z=\frac{x\sqrt{f}}{\sqrt{(1-gxx)}}$, $\sqrt{(f+gzz)}=\frac{\sqrt{f}}{\sqrt{(1-gxx)}}$ et $\sqrt{(h+kzz)}=\sqrt{\frac{h+(fk-gh)xx}{1-gxx}}$, unde conficitur

$$dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}}=\frac{fdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}},$$

$$dx\sqrt{\frac{h+(fk-gh)xx}{1-gxx}}=\frac{fdz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}.$$

Quare

$$\int\frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=\frac{1}{f}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}},$$

$$\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}}=\frac{1}{f}\int dx\sqrt{\frac{h+(fk-gh)xx}{1-gxx}}.$$

7. Simili modo ponendo $x=\frac{z}{\sqrt{(h+kzz)}}$ reperitur

$$\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=\frac{1}{h}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}},$$

$$\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}}=\frac{1}{h}\int dx\sqrt{\frac{f+(gh-fk)xx}{1-kxx}}.$$

8. Ponatur $x=\frac{\sqrt{(f+gzz)}}{z}$; erit $dx=\frac{-fdz}{zz\sqrt{(f+gzz)}}$, tum $z=\frac{\sqrt{f}}{\sqrt{(xx-g)}}$, $\sqrt{(f+gzz)}=\frac{\sqrt{f}x}{\sqrt{(xx-g)}}$ atque $\sqrt{(h+kzz)}=\sqrt{\frac{hxx+fk-gh}{xx-g}}$, unde fit

$$dx\sqrt{\frac{hxx+fk-gh}{xx-g}}=\frac{-fdz\sqrt{(h+kzz)}}{zz\sqrt{(f+gzz)}}$$

et

$$dx\sqrt{\frac{xx-g}{hxx+fk-gh}}=\frac{-fdz}{zz\sqrt{(f+gzz)(h+kzz)}},$$

unde deducitur

$$\int\frac{dz}{zz}\sqrt{\frac{h+kzz}{f+gzz}}=-\frac{1}{f}\int dx\sqrt{\frac{hxx+fk-gh}{xx-g}},$$

$$\int\frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}}=-\frac{1}{f}\int dx\sqrt{\frac{xx-g}{hxx+fk-gh}}.$$

9. Simili modo ponendo $x=\frac{\sqrt{(h+kzz)}}{z}$ reperietur

$$\int\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}=-\frac{1}{h}\int dx\sqrt{\frac{fxx-fk+gh}{xx-k}},$$

$$\int\frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}}=-\frac{1}{h}\int dx\sqrt{\frac{xx-k}{fxx-fk+gh}}.$$

10. Ponatur $x=\sqrt{\frac{f+gzz}{h+kzz}}$; erit $dx=\frac{(gh-fk)zdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}$, tum $z=\sqrt{\frac{hxx-f}{g-kxx}}$, unde colligitur

$$\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=\frac{1}{gh-fk}\int dx\sqrt{\frac{hxx-f}{g-kxx}},$$

$$\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=\frac{1}{gh-fk}\int dx\sqrt{\frac{g-kxx}{hxx-f}}.$$

11. Simili modo ponendo $x=\sqrt{\frac{h+kzz}{f+gzz}}$ reperitur

$$\int\frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=\frac{1}{fk-gh}\int dx\sqrt{\frac{fxx-h}{k-gxx}},$$

$$\int\frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=\frac{1}{fk-gh}\int dx\sqrt{\frac{k-gxx}{fxx-h}}.$$

COROLLARIUM 1

57. Formulas has in ordinem reducentes, quia quaelibet duplici modo ad formam canonicam reducitur, habebimus primo

$$\int\frac{dz}{zz}\sqrt{\frac{f+gzz}{h+kzz}}=-\int dx\sqrt{\frac{fxx+g}{hxx+k}}=-\frac{1}{h}\int dy\sqrt{\frac{fyy-fk+gh}{yy-k}}$$

existente

$$x=\frac{1}{z}\quad\text{et}\quad y=\frac{\sqrt{(h+kzz)}}{z},$$

$$\int\frac{dz}{zz}\sqrt{\frac{h+kzz}{f+gzz}}=-\int dx\sqrt{\frac{hxx+k}{fxx+g}}=-\frac{1}{f}\int dy\sqrt{\frac{hyy+fk-gh}{yy-g}}$$

existente

$$x=\frac{1}{z}\quad\text{et}\quad y=\frac{\sqrt{(f+gzz)}}{z}.$$

COROLLARIUM 2

58. Secunda forma haec esto

$$\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}=\frac{1}{g}\int dx\sqrt{\frac{xx-f}{gh-fk+kxx}}=\frac{1}{k}\int dy\sqrt{\frac{yy-h}{fk-gh+gyy}}$$

existente

$$x=\sqrt{(f+gzz)}\quad\text{et}\quad y=\sqrt{(h+kzz)},$$

quae permutatis formulis $\sqrt{(f+gzz)}$ et $\sqrt{(h+kzz)}$ non mutatur.

COROLLARIUM 3

59. Tertia forma ita constituatur

$$\int \frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} = -\frac{1}{k}\int dx\sqrt{\frac{g+(fk-gh)xx}{1-hxx}} = \frac{1}{h}\int dy\sqrt{\frac{f+(gh-fk)yy}{1-kyy}}$$

existente

$$x = \frac{1}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \frac{z}{\sqrt{(h+kzz)}},$$

$$\int \frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} = -\frac{1}{g}\int dx\sqrt{\frac{k+(gh-fk)xx}{1-fxx}} = \frac{1}{f}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}}$$

existente

$$x = \frac{1}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \frac{z}{\sqrt{(f+gzz)}}.$$

COROLLARIUM 4

60. Quarta forma haec statuatur

$$\int \frac{dz}{zz\sqrt{(f+gzz)(h+kzz)}} = -\frac{1}{f}\int dx\sqrt{\frac{xx-g}{hxx+fk-gh}} = -\frac{1}{h}\int dy\sqrt{\frac{yy-k}{fyy-fk+gh}}$$

existente

$$x = \frac{\sqrt{(f+gzz)}}{z} \quad \text{et} \quad y = \frac{\sqrt{(h+kzz)}}{z}.$$

COROLLARIUM 5

61. Quinta forma erit geminata

$$\int \frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} = \frac{1}{f}\int dx\sqrt{\frac{1-gxx}{h+(fk-gh)xx}} = \frac{1}{fk-gh}\int dy\sqrt{\frac{k-gyy}{fyy-h}}$$

existente

$$x = \frac{z}{\sqrt{(f+gzz)}} \quad \text{et} \quad y = \sqrt{\frac{h+kzz}{f+gzz}},$$

$$\int \frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} = \frac{1}{h}\int dx\sqrt{\frac{1-kxx}{f+(gh-fk)xx}} = \frac{1}{gh-fk}\int dy\sqrt{\frac{g-kyy}{hyy-f}}$$

existente

$$x = \frac{z}{\sqrt{(h+kzz)}} \quad \text{et} \quad y = \sqrt{\frac{f+gzz}{h+kzz}}.$$

COROLLARIUM 6

62. Sexta denique forma erit

$$\int\frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}}=-\frac{1}{g}\int dx\sqrt{\frac{1-fxx}{k+(gh-fk)xx}}=\frac{1}{fk-gh}\int dy\sqrt{\frac{fyy-h}{k-gyy}}$$

existente

$$x=\frac{1}{\sqrt{(f+gzz)}}\quad\text{et}\quad y=\sqrt{\frac{h+kzz}{f+gzz}},$$

$$\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}=-\frac{1}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}}=\frac{1}{gh-fk}\int dy\sqrt{\frac{hyy-f}{g-kyy}}$$

existente

$$x=\frac{1}{\sqrt{(h+kzz)}}\quad\text{et}\quad y=\sqrt{\frac{f+gzz}{h+kzz}}.$$

PROBLEMA 6

63. *Invenire casus, quibus expressio* $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ *aequatur quantitati algebraicae* $\alpha z\sqrt{\frac{f+gzz}{h+kzz}}$ *una cum arcu sectionis conicae.*

SOLUTIO

Ponatur $\int dz\sqrt{\frac{f+gzz}{h+kzz}}=\alpha z\sqrt{\frac{f+gzz}{h+kzz}}+Z$ eritque differentiando

$$dZ=\frac{dz\big((1-\alpha)fh+(fk+(1-2\alpha)gh)zz+(1-\alpha)gkz^4\big)}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}},$$

ubi numerator neque per $f+gzz$ neque per $h+kzz$ reddi potest divisibilis, quin simul fiat $fk=gh$; reducetur autem Z ad formam posteriorem § 62 ponendo $\alpha=1$ eritque

$$Z=(fk-gh)\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}.$$

Hinc habebimus vel

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}=z\sqrt{\frac{f+gzz}{h+kzz}}+\frac{gh-fk}{k}\int dx\sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$$

existente

$$x=\frac{1}{\sqrt{(h+kzz)}}$$

vel etiam

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = z\sqrt{\frac{f+gzz}{h+kzz}} - \int dy \sqrt{\frac{hyy-f}{g-kyy}}$$

existente

$$y = \sqrt{\frac{f+gzz}{h+kzz}}.$$

COROLLARIUM 1

64. Quoties ergo vel formula $\int dx \sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$ vel haec $\int dy \sqrt{\frac{hyy-f}{g-kyy}}$ ad quempiam casuum iam tractatorum referri potest, toties quoque formula $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ partim quantitati algebraicae partim arcui sectionis conicae aequabitur.

COROLLARIUM 2

65. Cum sit $x = \frac{1}{\sqrt{(h+kzz)}}$, erit $1-hxx = kxxzz$; ergo nisi sit k quantitas positiva, formula prior non ita, ut fecimus, repraesentari potest. Scilicet si k sit quantitas negativa, ita scribi debet

$$\int dx \sqrt{\frac{hxx-1}{(gh-fk)xx-g}}.$$

COROLLARIUM 3

66. In altera formula $\int dy \sqrt{\frac{hyy-f}{g-kyy}}$, ubi $y = \sqrt{\frac{f+gzz}{h+kzz}}$, quia est $hyy-f = \frac{(gh-fk)zz}{h+kzz}$, sumitur $gh > fk$. Quare si fuerit $gh < fk$, ea ita scribi debet $\int dy \sqrt{\frac{f-hyy}{-g+kyy}}$. Prior scriptio ergo locum habet, si $gh-fk > 0$, posterior vero, si $fk-gh > 0$.

EXEMPLUM 1

67. Reducatur forma $\int dx \sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$ ad casum III esseque oportet $k > 0$, $h < 0$, $g > 0$ et $fk-gh < 0$, unde $f < 0$, habebiturque

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}} = z\sqrt{\frac{-f+gzz}{-h+kzz}} + \frac{fk-gh}{k}\int dx \sqrt{\frac{1+hxx}{g-(fk-gh)xx}},$$

ubi esse debet $fk > gh$. Iam per § 40 erit

$$\int dx \sqrt{\frac{1+hxx}{g-(fk-gh)xx}} = C - \frac{fk}{(fk-gh)^{\frac{3}{2}}} \mathit{\Pi} \frac{fk-gh}{fk}\left(1 - x\sqrt{\frac{fk-gh}{g}}\right)\left[\frac{fk-gh}{fk}\right]$$

vel per § 53

$$\int dx \sqrt{\frac{1+hxx}{g-(fk-gh)xx}} = C + \frac{1}{\sqrt{fk}} \mathit{\Pi} \frac{fk}{fk-gh}\left(1 - \frac{\sqrt{(g-(fk-gh)xx)}}{\sqrt{g}}\right)\left[\frac{fk}{fk-gh}\right],$$

ubi est

$$x = \frac{1}{\sqrt{(-h+kzz)}} \quad \text{et} \quad \sqrt{(g-(fk-gh)xx)} = \frac{\sqrt{k}\,(gzz-f)}{\sqrt{(-h+kzz)}};$$

sicque construitur casus XI.

INTEGRATIO CASUS XI

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{-h+kzz}} - \frac{f}{\sqrt{(fk-gh)}} \mathit{\Pi} \frac{fk-gh}{fk}\left(1 - \frac{\sqrt{(fk-gh)}}{\sqrt{g(-h+kzz)}}\right)\left[\frac{fk-gh}{fk}\right]$$

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{-h+kzz}} + \frac{fk-gh}{k\sqrt{fk}} \mathit{\Pi} \frac{fk}{fk-gh}\left(1 - \frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-h+kzz)}}\right)\left[\frac{fk}{fk-gh}\right]$$

68. Hoc ergo integrale constat parte algebraica et arcu elliptico, et quia debet esse $fk > gh$, fieri nequit $= 0$; sin autem sit $h = 0$, ellipsis abit in circulum atque habebitur

$$\int \frac{dz}{z} \sqrt{\frac{-f+gzz}{k}} = C + \sqrt{\frac{-f+gzz}{k}} - \frac{\sqrt{f}}{\sqrt{k}} \mathit{\Pi}\left(1 - \frac{\sqrt{f}}{z\sqrt{g}}\right)[1]$$

seu

$$\int \frac{dz}{z} \sqrt{\frac{-f+gzz}{k}} = C + \sqrt{\frac{-f+gzz}{k}} + \frac{\sqrt{f}}{\sqrt{k}} \mathit{\Pi}\left(1 - \frac{\sqrt{(-f+gzz)}}{z\sqrt{g}}\right)[1],$$

uti per integrationem facile invenitur.

EXEMPLUM 2

69. Reducatur formula $\int dx \sqrt{\frac{1-hxx}{g+(fk-gh)xx}}$ ad casum VI eritque $h>0$, $g>0$, $gh-fk>0$ et $k>0$; cum autem hoc casu debeat esse $gh-fk>gh$, quantitatem f negative capi oportet, ut sit posito $x=\frac{1}{\sqrt{(h+kzz)}}$

$$\int dx \sqrt{\frac{-f+gzz}{h+kzz}} = z\sqrt{\frac{-f+gzz}{h+kzz}} + \frac{gh+fk}{k}\int dx \sqrt{\frac{1-hxx}{g-(fk+gh)xx}}.$$

At ex § 41 habetur

$$\int dx \sqrt{\frac{1-hxx}{g-(fk+gh)xx}} = C - \frac{fk}{(fk+gh)^{\frac{3}{2}}}\, \Pi \frac{fk+gh}{fk}\left(1-x\sqrt{\frac{fk+gh}{g}}\right)\left[\frac{fk+gh}{fk}\right],$$

unde casus IX conficitur.

INTEGRATIO CASUS IX

$$\int dz \sqrt{\frac{-f+gzz}{h+kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{h+kzz}} - \frac{f}{\sqrt{(fk+gh)}}\, \Pi \frac{fk+gh}{fk}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{g(h+kzz)}}\right)\left[\frac{fk+gh}{fk}\right]$$

70. Casus ergo huius integrale constat parte algebraica et arcu elliptico, qui ut semper adhuc alio modo exprimi posset; verum praeferenda est illa ellipsis, cuius axis parametrum superat, ne certis casibus evanescere queat. Caeterum hunc casum ex praecedente XI derivare potuissemus ponendo h negativum, atque si in forma posteriori faciemus $gh>fk$, habebimus aliam integrationem casus XII.

INTEGRATIO CASUS XII

$$\int dz \sqrt{\frac{-f+gzz}{-h+kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{-h+kzz}} - \frac{gh-fk}{k\sqrt{fk}}\, \Pi \frac{fk}{gh-fk}\left(\frac{\sqrt{k(-f+gzz)}}{\sqrt{g(-h+kzz)}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

71. En aliam integrationem casus XII iam supra § 42 tractati, quae praeter arcum hyperbolicum continet partem algebraicam, cum prior solo

arcu hyperbolico exprimatur. Aequalitas ergo harum duarum expressionum perpendi meretur; quod quo concinnius fiat, ponamus $\frac{fk}{gh-fk}=\frac{m}{n}$ et $z\sqrt{\frac{k}{h}}=t$ eritque

$$\Pi\frac{m}{n}(t-1)\left[\frac{-m}{n}\right]+\Pi\frac{m}{n}\left(\sqrt{\frac{(m+n)tt-m}{(m+n)(tt-1)}}-1\right)\left[\frac{-m}{n}\right]$$
$$=C+\frac{m}{n}t\sqrt{\frac{(m+n)tt-m}{m(tt-1)}},$$

unde constante debite definita diversi arcus hyperbolici inter se comparari possunt. Scilicet posito semiaxe $\frac{m}{n}=\alpha$ sumtisque duabus variabilibus t et u erit

$$\left.\begin{aligned}&+\Pi\alpha(t-1)[-\alpha]+\Pi\alpha\left(\sqrt{\frac{(\alpha+1)tt-\alpha}{(\alpha+1)(tt-1)}}-1\right)[-\alpha]\\&-\Pi\alpha(u-1)[-\alpha]-\Pi\alpha\left(\sqrt{\frac{(\alpha+1)uu-\alpha}{(\alpha+1)(uu-1)}}-1\right)[-\alpha]\end{aligned}\right\}=\left\{\begin{aligned}&+\alpha t\sqrt{\frac{(\alpha+1)tt-\alpha}{\alpha(tt-1)}}\\&-\alpha u\sqrt{\frac{(\alpha+1)uu-\alpha}{\alpha(uu-1)}}.\end{aligned}\right.$$

EXEMPLUM 3

72. Ponamus f et k negativa et posterior expressio dat

$$\int dz\sqrt{\frac{-f+gzz}{h-kzz}}=z\sqrt{\frac{-f+gzz}{h-kzz}}-\int dy\sqrt{\frac{f+hyy}{g+kyy}}$$

existente $gh>fk$. Iam ex casu II § 51 tractato habemus

$$\int dy\sqrt{\frac{f+hyy}{g+kyy}}=C+\frac{f}{\sqrt{(gh-fk)}}\Pi\frac{gh-fk}{fk}\left(\frac{\sqrt{(g+kyy)}}{\sqrt{g}}-1\right)\left[\frac{-gh+fk}{fk}\right].$$

Cum igitur sit

$$y=\sqrt{\frac{-f+gzz}{h-kzz}},\quad\text{erit}\quad\sqrt{(g+kyy)}=\frac{\sqrt{(gh-fk)}}{\sqrt{(h-kzz)}},$$

unde casus X expeditur.

INTEGRATIO CASUS X

$$\int dz \sqrt{\frac{-f+gzz}{h-kzz}}$$

$$= C + z\sqrt{\frac{-f+gzz}{h-kzz}} - \frac{f}{\sqrt{(gh-fk)}}\,\Pi\,\frac{gh-fk}{fk}\left(\frac{\sqrt{(gh-fk)}}{\sqrt{g(h-kzz)}}-1\right)\left[\frac{-gh+fk}{fk}\right]$$

73. Huius ergo casus X integrale constat parte algebraica et arcu hyperbolico. Sin autem k sumatur negative, oritur integrale casus IX iam ante § 70 exhibitum, ex quo hic ipse casus derivari potuisset.

EXEMPLUM 4

74. Capiantur g et k negative, ut sit $y = \sqrt{\frac{f-gzz}{h-kzz}}$, eritque

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}} = z\sqrt{\frac{f-gzz}{h-kzz}} - \int dy \sqrt{\frac{hyy-f}{kyy-g}}.$$

Quodsi forma $\int dy \sqrt{\frac{hyy-f}{kyy-g}}$ hoc modo repraesentetur, ob g et k negative sumtas debet esse $fk - gh > 0$, tum autem non in casu XII continetur, verum hoc modo $\int dy \sqrt{\frac{f-hyy}{g-kyy}}$ repraesentata exigit $gh > fk$, quae conditio casui VI, quorsum esset referenda, adversatur.

SCHOLION

75. Ope ergo praecedentis problematis casus IX, X et XI sumus executi, cum ante iam casus III, VI et XII, tum vero etiam II per simplices arcus expediverimus. Restant ergo quinque casus nondum realiter resoluti, quorum nonnullos ita tractare poterimus, ut integrale constet arcu sectionis conicae et quantitate algebraica formae $z\sqrt{\frac{h+kzz}{f+gzz}}$.

PROBLEMA 7

76. *Invenire casus, quibus expressio* $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ *aequatur quantitati algebraicae* $\alpha z\sqrt{\frac{h+kzz}{f+gzz}}$ *una cum arcu sectionis conicae.*

SOLUTIO

Ponatur

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = \alpha z\sqrt{\frac{h+kzz}{f+gzz}} + Z;$$

erit differentiando

$$dZ = \frac{dz(ff - \alpha fh + 2f(g - \alpha k)zz + g(g - \alpha k)z^4)}{(f + gzz)^{\frac{3}{2}}\sqrt{(h + kzz)}},$$

ubi notandum est numeratorem per $f + gzz$ reddi non posse divisibilem, quin simul α evanescat. At si ad quandam superiorum formularum reducere velimus, poni oportet $\alpha = \frac{g}{k}$, quo facto oritur

$$dZ = \frac{f(fk - gh)}{k} \cdot \frac{dz}{(f + gzz)^{\frac{3}{2}}\sqrt{(h + kzz)}},$$

cuius integratio per § 61 constat. Habebimus ergo vel

$$\int dz\sqrt{\frac{f + gzz}{h + kzz}} = C + \frac{g}{k}z\sqrt{\frac{h + kzz}{f + gzz}} + \frac{fk - gh}{k}\int dx\sqrt{\frac{1 - gxx}{h + (fk - gh)xx}}$$

existente

$$x = \frac{z}{\sqrt{(f + gzz)}}$$

vel

$$\int dz\sqrt{\frac{f + gzz}{h + kzz}} = C + \frac{g}{k}z\sqrt{\frac{h + kzz}{f + gzz}} + \frac{f}{k}\int dy\sqrt{\frac{k - gyy}{fyy - h}}$$

existente

$$y = \sqrt{\frac{h + kzz}{f + gzz}}.$$

COROLLARIUM 1

77. Cum sit $x = \frac{z}{\sqrt{(f + gzz)}}$, erit

$$1 - gxx = \frac{fxx}{zz};$$

quare si fuerit f quantitas positiva, formula recte hoc modo

$$\int dx\sqrt{\frac{1 - gxx}{h + (fk - gh)xx}}$$

exprimitur; sin autem sit $f < 0$, ita debet repraesentari

$$\int dx\sqrt{\frac{gxx - 1}{(gh - fk)xx - h}}.$$

COROLLARIUM 2

78. Cum sit $y = \sqrt{\frac{h+kzz}{f+gzz}}$, erit

$$fyy - h = \frac{(fk-gh)zz}{f+gzz},$$

unde, si formula integralis ita exhibeatur

$$\int dy \sqrt{\frac{k-gyy}{fyy-h}},$$

neccesse est sit $fk - gh > 0$; sin autem ita exprimatur

$$\int dy \sqrt{\frac{-k+gyy}{h-fyy}},$$

oportet sit $gh - fk > 0$.

EXEMPLUM 1

79. Referatur forma

$$\int dx \sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

ad casum III, et quia est $f > 0$, sumi debet $g < 0$, $h > 0$ et $k < 0$, unde obtinetur

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}} = C + \frac{g}{k} z \sqrt{\frac{h-kzz}{f-gzz}} + \frac{fk-gh}{k} \int dx \sqrt{\frac{1+gxx}{h-(fk-gh)xx}};$$

est vero ex casu III (§ 40)

$$\int dx \sqrt{\frac{1+gxx}{h-(fk-gh)xx}} = \frac{-fk}{fk-gh} \sqrt{\frac{1}{fk-gh}}\, \Pi \frac{fk-gh}{fk} \left(1 - x\sqrt{\frac{fk-gh}{h}}\right) \left[\frac{fk-gh}{fk}\right],$$

ubi, cum sit $fk > gh$, iterum casus VI occurrit.

INTEGRATIO CASUS VI

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}}$$

$$= C + \frac{g}{k} z \sqrt{\frac{h-kzz}{f-gzz}} - \frac{f}{\sqrt{(fk-gh)}}\, \Pi \frac{fk-gh}{fk} \left(1 - \frac{z\sqrt{(fk-gh)}}{\sqrt{h(f-gzz)}}\right) \left[\frac{fk-gh}{fk}\right]$$

80. Si ellipsin in aliam sui similem invertamus, erit

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}}$$

$$= C + \frac{g}{k} z \sqrt{\frac{h-kzz}{f-gzz}} + \frac{fk-gh}{k\sqrt{fk}}\, \Pi \frac{fk}{fk-gh} \left(1 - \frac{\sqrt{f(h-kzz)}}{\sqrt{h(f-gzz)}}\right) \left[\frac{fk}{fk-gh}\right],$$

quod integrale cum superiori § 41 comparatum egregiam suppeditat arcuum ellipticorum relationem. Sit autem semiaxis

$$\frac{fk}{fk-gh}=a \quad \text{et} \quad z\sqrt{\frac{k}{h}}=t \quad \text{seu} \quad zz=\frac{htt}{k};$$

erit

$$\sqrt{\frac{h-kzz}{f-gzz}}=\sqrt{\frac{b}{f}}\cdot\frac{a(1-tt)}{a-(a-1)tt}$$

ob $gh=\frac{a-1}{a}fk$, unde fit

$$\varPi a(1-t)[a]+\varPi a\left(1-\sqrt{\frac{a(1-tt)}{a-(a-1)tt}}\right)[a]+(a-1)t\sqrt{\frac{a(1-tt)}{a-(a-1)tt}}=C.$$

Sumtis ergo duabus variabilibus t et u habebitur

$$\left.\begin{aligned}&+\varPi a(1-t)[a]+\varPi a\left(1-\sqrt{\frac{a(1-tt)}{a-(a-1)tt}}\right)[a]\\&-\varPi a(1-u)[a]-\varPi a\left(1-\sqrt{\frac{a(1-uu)}{a-(a-1)uu}}\right)[a]\end{aligned}\right\}=\left\{\begin{aligned}&-(a-1)t\sqrt{\frac{a(1-tt)}{a-(a-1)tt}}\\&+(a-1)u\sqrt{\frac{a(1-uu)}{a-(a-1)uu}}\end{aligned}\right.$$

unde comparationes arcuum ellipticorum dudum a me demonstratae facile colliguntur.

Si hic sumatur g negative, oritur casus III et tum formula

$$\int dx\sqrt{\frac{1-gxx}{h-(fk+gh)xx}}$$

ad casum VI referenda fuisset, quare non opus est, ut hunc casum evolvamus.

EXEMPLUM 2

81. Haec forma nisi invertatur,

$$\int dx\sqrt{\frac{gxx-1}{(gh-fk)xx-h}}$$

ad casum XII reduci nequit, ubi esse debet $f<0$; habebimus ergo

$$\int dz\sqrt{\frac{-f+gzz}{h+kzz}}=C+\frac{g}{k}z\sqrt{\frac{h+kzz}{-f+gzz}}-\frac{fk+gh}{k}\int dx\sqrt{\frac{gxx-1}{(fk+gh)xx-h}},$$

verum nunc ad casum XI refertur indeque acquireremus casum IX iam supra inventum.

EXEMPLUM 3

82. Verum formulam

$$\int dx \sqrt{\frac{1-gxx}{h+(fk-gh)xx}}$$

ad casum II reducamus, quod fit sumendo $g<0$ existente $f>0$ et

$$x=\frac{z}{\sqrt{(f-gzz)}},$$

ut habeatur

$$\int dz \sqrt{\frac{f-gzz}{h+kzz}}=C-\frac{g}{k}z\sqrt{\frac{h+kzz}{f-gzz}}+\frac{fk+gh}{k}\int dx \sqrt{\frac{1+gxx}{h+(fk+gh)xx}};$$

verum haec reductio non succedit, nisi $k<0$, ita ut sit

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}}=C+\frac{g}{k}z\sqrt{\frac{h-kzz}{f-gzz}}-\frac{gh-fk}{k}\int dx \sqrt{\frac{1+gxx}{h+(gh-fk)xx}}$$

existente

$$x=\frac{z}{\sqrt{(f-gzz)}},$$

eritque ex § 51

$$\int dx \sqrt{\frac{1+gxx}{h+(gh-fk)xx}}=\frac{1}{\sqrt{fk}}\,\Pi\,\frac{fk}{gh-fk}\left(\frac{\sqrt{(h+(gh-fk)xx)}}{\sqrt{h}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

et

$$\sqrt{(h+(gh-fk)xx)}=\frac{\sqrt{f(h-kzz)}}{\sqrt{(f-gzz)}},$$

unde casus VII colligitur.

INTEGRATIO CASUS VII

$$\int dz \sqrt{\frac{f-gzz}{h-kzz}}$$

$$=C+\frac{g}{k}z\sqrt{\frac{h-kzz}{f-gzz}}-\frac{gh-fk}{k\sqrt{fk}}\,\Pi\,\frac{fk}{gh-fk}\left(\frac{\sqrt{f(h-kzz)}}{\sqrt{h(f-gzz)}}-1\right)\left[\frac{-fk}{gh-fk}\right]$$

EXISTENTE $gh>fk$

83. Constat ergo hoc integrale parte algebraica et arcu hyperbolico hicque casus ad iam expeditos de novo accedit.

SCHOLION

84. Hactenus ergo octo casus per valores reales integravimus, qui sunt II, III, VI, VII, IX, X, XI et XII, et reliqui quatuor ita sunt comparati, ut per similes formas nullo modo integrari queant. Exigunt scilicet praeter partem algebraicam duos arcus, alterum ellipticum, alterum hyperbolicum, ac pars quidem algebraica vel huius formae $z\sqrt{\frac{f+gzz}{h+kzz}}$ vel huius $z\sqrt{\frac{h+kzz}{f+gzz}}$ assumi potest; unde duo adhuc problemata evolvi conveniet.

PROBLEMA 8

85. *Invenire casus, quibus expressio* $\int dz\sqrt{\frac{f+gzz}{h+kzz}}$ *aequatur quantitati algebraicae* $\alpha z\sqrt{\frac{f+gzz}{h+kzz}}$ *una cum duobus arcubus sectionum conicarum.*

SOLUTIO

Posito

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}} = \alpha z\sqrt{\frac{f+gzz}{h+kzz}} + Z$$

erit differentiando

$$dZ = \frac{dz\big((1-\alpha)fh + (fk+(1-2\alpha)gh)zz + (1-\alpha)gkz^4\big)}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}},$$

quae in duas partes formulis probl. 5. traditis contentas resolvantur.

1. Ponatur

$$Z = p\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int\frac{zz\,dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}}$$

fierique debet

$$(1-\alpha)fh = p, \quad fk+(1-2\alpha)gh = q, \quad (1-\alpha)gk = 0,$$

unde ob $\alpha = 1$ evanesceret quoque p contra hypothesin.

2. Ponatur

$$Z = p\int\frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}}$$

fierique debet

$$(1-\alpha)fh = p+qf, \quad fk+(1-2\alpha)gh = qg \quad \text{et} \quad (1-\alpha)gk = 0,$$

unde

$$\alpha = 1, \quad q = \frac{fk-gh}{g}, \quad p = \frac{-f(fk-gh)}{g}$$

ideoque

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C + z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{g}\int dy \sqrt{\frac{g-kyy}{hyy-f}} - \frac{fk-gh}{gk}\int dx \sqrt{\frac{g+(fk-gh)xx}{1-hxx}}$$

existente

$$y = \sqrt{\frac{f+gzz}{h+kzz}} \quad \text{et} \quad x = \frac{1}{\sqrt{(h+kzz)}}.$$

3. Ponatur

$$Z = p\int \frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int \frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}$$

ac fieri necesse est

$$(1-\alpha)fh = p, \quad fk + (1-2\alpha)gh = qh, \quad (1-\alpha)gk = qk,$$

unde deducitur

$$\alpha = \frac{fk}{gh}, \quad q = \frac{gh-fk}{h}, \quad p = \frac{f(gh-fk)}{g}.$$

Quocirca habebimus

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C + \frac{fk}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{g}\int dy \sqrt{\frac{g-kyy}{hyy-f}} + \frac{gh-fk}{gh}\int dx \sqrt{\frac{xx-f}{gh-fk+kxx}}$$

existente

$$y = \sqrt{\frac{f+gzz}{h+kzz}} \quad \text{et} \quad x = \sqrt{(f+gzz)}.$$

4. Ponatur

$$Z = p\int \frac{dz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int dz \sqrt{\frac{h+kzz}{f+gzz}}$$

fierique oportet

$$(1-\alpha)fh = p + qhh, \quad fk + (1-2\alpha)gh = 2qhk, \quad (1-\alpha)gk = qkk,$$

unde deducitur $fk - gh = 0$, quod est absurdum.

5. Ponatur

$$Z = p\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}};$$

fiet

$$(1-\alpha)fh = qf, \quad fk + (1-2\alpha)gh = p + qg \quad \text{et} \quad (1-\alpha)gk = 0,$$

unde nihil ob $q = 0$ concludere licet.

6. Ponatur

$$Z = p\int\frac{zzdz}{(h+kzz)^{\frac{3}{2}}\sqrt{(f+gzz)}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}}$$

fietque

$$(1-\alpha)fh = qhh, \quad fk + (1-2\alpha)gh = p + 2qhk, \quad (1-\alpha)gk = qkk,$$

unde pariter nihil colligi potest.

7. Ponatur

$$Z = p\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} + q\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}$$

eritque

$$(1-\alpha)fh = pf, \quad fk + (1-2\alpha)gh = pg + qh, \quad (1-\alpha)gk = qk,$$

unde quoque nihil concluditur.

8. Ponatur

$$Z = p\int\frac{dz\sqrt{(f+gzz)}}{(h+kzz)^{\frac{3}{2}}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}}$$

eritque

$$(1-\alpha)fh = pf + qhh, \quad fk + (1-2\alpha)gh = pg + 2qhk, \quad (1-\alpha)gk = qkk,$$

unde elicitur

$$\alpha = \frac{gh-fk}{gh}, \quad p = \frac{fk-gh}{g}, \quad q = \frac{f}{h}.$$

Quare erit

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C + \frac{gh-fk}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{gh}\int dy\sqrt{\frac{f-(fk-gh)yy}{1-kyy}}$$

existente

$$y = \frac{z}{\sqrt{(h+kzz)}}.$$

Plures combinationes idoneas instituere non licet.

COROLLARIUM 1

86. Ex hypothesi ultima sponte sequitur integratio casus I, quo est $fk > gh$; ex casu enim II est

$$\int dz\sqrt{\frac{h+kzz}{f+gzz}} = \frac{h}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right],$$

deinde ex casu VI est (§ 41)

$$\int dy\sqrt{\frac{f-(fk-gh)yy}{1-kyy}} = \frac{-gh}{fk}\sqrt{\frac{f}{k}}\,\Pi\frac{fk}{gh}(1-y\sqrt{k})\left[\frac{fk}{gh}\right]$$

hincque colligitur

INTEGRATIO CASUS I

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C - \frac{fk-gh}{gh}z\sqrt{\frac{f+gzz}{h+kzz}} + \frac{f}{\sqrt{(fk-gh)}}\,\Pi\frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$
$$- \frac{fk-gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(1-\frac{z\sqrt{k}}{\sqrt{(h+kzz)}}\right)\left[\frac{fk}{gh}\right].$$

COROLLARIUM 2

87. Ex hypothesi n° 3 casus V deduci posse videtur, unde fit

$$\int dz\sqrt{\frac{f-gzz}{h+kzz}} = \frac{-fk}{gh}z\sqrt{\frac{f-gzz}{h+kzz}} - \frac{f}{g}\int dy\sqrt{\frac{g+kyy}{f-hyy}} + \frac{fk+gh}{gh}\int dx\sqrt{\frac{f-xx}{fk+gh-kxx}}$$

existente

$$y = \sqrt{\frac{f-gzz}{h+kzz}} \quad \text{et} \quad x = \sqrt{(f-gzz)};$$

sed haec ultima formula ex casu VI confici nequit neque etiam ex hypothesi n° 2.

COROLLARIUM 3

88. Consideremus formam VIII, ubi g et h sunt negativa, $fk > gh$, atque n° 3 huc transferendo habebimus

$$\int dz\sqrt{\frac{f-gzz}{-h+kzz}} = \frac{fk}{gh}z\sqrt{\frac{f-gzz}{-h+kzz}} - \frac{f}{g}\int dy\sqrt{\frac{g+kyy}{f+hyy}} + \frac{gh-fk}{gh}\int dx\sqrt{\frac{f-xx}{fk-gh-kxx}}$$

existente

$$y = \sqrt{\frac{f - gzz}{-h + kzz}} \quad \text{et} \quad x = \sqrt{(f - gzz)};$$

nunc vero est ex casu II

$$\int dy \sqrt{\frac{g + kyy}{f + hyy}} = \frac{g}{\sqrt{(fk - gh)}} \, \Pi \, \frac{fk - gh}{gh} \left(\frac{\sqrt{(f + hyy)}}{\sqrt{f}} - 1 \right) \left[\frac{-fk + gh}{gh} \right]$$

existente

$$\sqrt{(f + hyy)} = \frac{z\sqrt{(fk - gh)}}{\sqrt{(-h + kzz)}},$$

deinde ex casu VI

$$\int dx \sqrt{\frac{f - xx}{fk - gh - kxx}} = \frac{-gh}{k\sqrt{fk}} \, \Pi \, \frac{fk}{gh} \left(1 - x\sqrt{\frac{k}{fk - gh}} \right) \left[\frac{fk}{gh} \right],$$

unde sequitur

INTEGRATIO CASUS VIII

$$\int dz \sqrt{\frac{f - gzz}{-h + kzz}}$$

$$= C + \frac{fk}{gh} z \sqrt{\frac{f - gzz}{-h + kzz}} - \frac{f}{\sqrt{(fk - gh)}} \, \Pi \, \frac{fk - gh}{gh} \left(\frac{z\sqrt{(fk - gh)}}{\sqrt{f(-h + kzz)}} - 1 \right) \left[\frac{-fk + gh}{gh} \right]$$

$$+ \frac{fk - gh}{k\sqrt{fk}} \, \Pi \, \frac{fk}{gh} \left(1 - \frac{\sqrt{k(f - gzz)}}{\sqrt{(fk - gh)}} \right) \left[\frac{fk}{gh} \right].$$

SCHOLION

89. Sic igitur casus duos novos I et VIII sumus adepti, ita ut tantum IV et V supersint, quos ope sequentis problematis superare licebit.

PROBLEMA 9

90. *Invenire casus, quibus expressio* $\int dz \sqrt{\frac{f + gzz}{h + kzz}}$ *aequatur quantitati algebraicae* $\alpha z \sqrt{\frac{h + kzz}{f + gzz}}$ *una cum duobus arcubus sectionum conicarum.*

SOLUTIO

Posito

$$\int dz \sqrt{\frac{f + gzz}{h + kzz}} = \alpha z \sqrt{\frac{h + kzz}{f + gzz}} + Z$$

erit differentiando

$$dZ = \frac{dz(ff - \alpha fh + 2f(g - \alpha k)zz + g(g - \alpha k)z^4)}{(f + gzz)^{\frac{3}{2}} \sqrt{(h + kzz)}},$$

cuius resolutio in duas partes idoneas sequenti modo instituatur.

1. Ponatur

$$Z = p \int \frac{zz\,dz}{(f + gzz)^{\frac{3}{2}} \sqrt{(h + kzz)}} + q \int \frac{dz \sqrt{(h + kzz)}}{(f + gzz)^{\frac{3}{2}}}$$

fietque

$$f(f - \alpha h) = qh, \quad 2f(g - \alpha k) = p + qk, \quad g(g - \alpha k) = 0,$$

unde colligitur

$$\alpha = \frac{g}{k}, \quad q = \frac{f(fk - gh)}{hk}, \quad p = \frac{-f(fk - gh)}{h}$$

ac propterea

$$\int dz \sqrt{\frac{f + gzz}{h + kzz}}$$

$$= C + \frac{gz}{k} \sqrt{\frac{h + kzz}{f + gzz}} - \frac{f}{h} \int dy \sqrt{\frac{fyy - h}{k - gyy}} + \frac{fk - gh}{hk} \int dx \sqrt{\frac{h + (fk - gh)xx}{1 - gxx}}$$

existente

$$y = \sqrt{\frac{h + kzz}{f + gzz}} \quad \text{et} \quad x = \frac{z}{\sqrt{(f + gzz)}}.$$

2. Ponatur

$$Z = p \int \frac{zz\,dz}{(f + gzz)^{\frac{3}{2}} \sqrt{(h + kzz)}} + q \int \frac{zz\,dz}{\sqrt{(f + gzz)(h + kzz)}}$$

fietque

$$f(f - \alpha h) = 0, \quad 2f(g - \alpha k) = p + qf, \quad g(g - \alpha k) = qg$$

hincque

$$\alpha = \frac{f}{h}, \quad p = \frac{f(gh - fk)}{h} \quad \text{et} \quad q = \frac{gh - fk}{h};$$

quare habebitur

$$\int dz \sqrt{\frac{f + gzz}{h + kzz}} = C + \frac{fz}{h} \sqrt{\frac{h + kzz}{f + gzz}} - \frac{f}{h} \int dy \sqrt{\frac{fyy - h}{k - gyy}} + \frac{gh - fk}{gh} \int dx \sqrt{\frac{xx - f}{gh - fk + kxx}}$$

existente

$$y = \sqrt{\frac{h + kzz}{f + gzz}} \quad \text{et} \quad x = \sqrt{(f + gzz)}.$$

3. Ponatur

$$Z = p\int\frac{zzdz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}}$$

fietque

$$f(f-\alpha h) = qfh, \quad 2f(g-\alpha k) = p + q(fk+gh), \quad g(g-\alpha k) = qgk,$$

unde nihil concludere licet.

4. Ponatur

$$Z = p\int\frac{dz}{(f+gzz)^{\frac{3}{2}}\sqrt{(h+kzz)}} + q\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}$$

fietque

$$f(f-\alpha h) = p, \quad 2f(g-\alpha k) = qf, \quad g(g-\alpha k) = qg,$$

unde nihil concludere licet.

5. Ponatur

$$Z = p\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} + q\int\frac{zzdz}{\sqrt{(f+gzz)(h+kzz)}}$$

fietque

$$f(f-\alpha h) = ph, \quad 2f(g-\alpha k) = pk + qf, \quad g(g-\alpha k) = qg,$$

unde nihil colligere licet.

6. Ponatur

$$Z = p\int\frac{dz\sqrt{(h+kzz)}}{(f+gzz)^{\frac{3}{2}}} + q\int dz\sqrt{\frac{h+kzz}{f+gzz}};$$

fieri debet

$$f(f-\alpha h) = ph + qfh, \quad 2f(g-\alpha k) = pk + q(fk+gh), \quad g(g-\alpha k) = qgk,$$

unde colligitur

$$\alpha = \frac{gh-fk}{hk}, \quad p = \frac{f(fk-gh)}{hk} \quad \text{et} \quad q = \frac{f}{h}$$

ideoque

$$\int dz\sqrt{\frac{f+gzz}{h+kzz}}$$

$$= C + \frac{gh-fk}{hk}z\sqrt{\frac{h+kzz}{f+gzz}} + \frac{fk-gh}{hk}\int dy\sqrt{\frac{h+(fk-gh)yy}{1-gyy}} + \frac{f}{h}\int dz\sqrt{\frac{h+kzz}{f+gzz}}$$

existente

$$y = \frac{z}{\sqrt{(f+gzz)}}.$$

COROLLARIUM 1

91. Hinc omnes quatuor casus difficiliores derivari possunt. Primus nempe statim deducitur ex n° 6; nam ob $fk > gh$ erit ex casu III

$$\int dy \sqrt{\frac{h+(fk-gh)yy}{1-gyy}} = -\frac{fk}{g\sqrt{gh}}\, \Pi \frac{gh}{fk}(1-y\sqrt{g})\left[\frac{gh}{fk}\right]$$

existente

$$y = \frac{z}{\sqrt{(f+gzz)}},$$

tum vero ex casu II

$$\int dz \sqrt{\frac{h+kzz}{f+gzz}} = \frac{h}{\sqrt{(fk-gh)}}\, \Pi \frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right] \text{ [1]}$$

hincque

INTEGRATIO CASUS I

$$\int dz \sqrt{\frac{f+gzz}{h+kzz}} = C - \frac{fk-gh}{hk} z\sqrt{\frac{h+kzz}{f+gzz}} - \frac{f(fk-gh)}{gh\sqrt{gh}}\, \Pi \frac{gh}{fk}\left(1-\frac{z\sqrt{g}}{\sqrt{(f+gzz)}}\right)\left[\frac{gh}{fk}\right]$$
$$+ \frac{f}{\sqrt{(fk-gh)}}\, \Pi \frac{fk-gh}{gh}\left(\frac{\sqrt{(f+gzz)}}{\sqrt{f}}-1\right)\left[\frac{-fk+gh}{gh}\right] \text{ [1]}.$$

COROLLARIUM 2

92. Hic membrum medium per inversionem ellipsis abit in

$$+\frac{fk-gh}{k\sqrt{fk}}\, \Pi \frac{fk}{gh}\left(1-\frac{\sqrt{f}}{\sqrt{(f+gzz)}}\right)\left[\frac{fk}{gh}\right],$$

unde si g negative capiatur, pro casu V manifesto fit pro hyperbola. At sumto g negativo erit ultimum membrum ex casu III

$$\int dz \sqrt{\frac{h+kzz}{f-gzz}} = \frac{-(fk+gh)}{gh}\sqrt{\frac{h}{g}}\, \Pi \frac{gh}{fk+gh}\left(1-z\sqrt{\frac{g}{f}}\right)\left[\frac{gh}{fk+gh}\right]$$
$$= +\frac{h}{\sqrt{(fk+gh)}}\, \Pi \frac{fk+gh}{gh}\left(1-\frac{\sqrt{(f-gzz)}}{\sqrt{f}}\right)\left[\frac{fk+gh}{gh}\right],$$

unde deducitur

---

1) Editio princeps: $\Pi \frac{fk-gh}{gh}\left(\frac{\sqrt{(h+kzz)}}{\sqrt{h}}-1\right)\left[\frac{-fk+gh}{gh}\right]$. Correxit A. K.

INTEGRATIO CASUS V

$$\int dz\sqrt{\frac{f-gzz}{h+kzz}}=C-\frac{fk+gh}{hk}z\sqrt{\frac{h+kzz}{f-gzz}}+\frac{fk+gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(\frac{\sqrt{f}}{\sqrt{(f-gzz)}}-1\right)\left[\frac{-fk}{gh}\right]$$

$$+\frac{f}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{gh}\left(1-\frac{\sqrt{(f-gzz)}}{\sqrt{f}}\right)\left[\frac{fk+gh}{gh}\right].$$

COROLLARIUM 3

93. Per n° 2 construitur casus IV, quo h negative capitur. Erit enim

$$\int dz\sqrt{\frac{f+gzz}{-h+kzz}}$$

$$=C-\frac{fz}{h}\sqrt{\frac{-h+kzz}{f+gzz}}+\frac{f}{h}\int dy\sqrt{\frac{h+fyy}{k-gyy}}+\frac{fk+gh}{gh}\int dx\sqrt{\frac{-f+xx}{-fk-gh+kxx}}$$

existente

$$y=\sqrt{\frac{-h+kzz}{f+gzz}}\quad\text{et}\quad x=\sqrt{(f+gzz)}.$$

Nunc vero est

$$\int dy\sqrt{\frac{h+fyy}{k-gyy}}=\frac{-(fk+gh)}{gh}\sqrt{\frac{h}{g}}\,\Pi\frac{gh}{fk+gh}\left(1-y\sqrt{\frac{g}{k}}\right)\left[\frac{gh}{fk+gh}\right]$$

$$=+\frac{h}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{gh}\left(1-\frac{\sqrt{(k-gyy)}}{\sqrt{k}}\right)\left[\frac{fk+gh}{gh}\right]$$

existente

$$\sqrt{(k-gyy)}=\frac{\sqrt{(fk+gh)}}{\sqrt{(f+gzz)}}$$

et

$$\int dx\sqrt{\frac{-f+xx}{-fk-gh+kxx}}=\frac{gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(x\sqrt{\frac{k}{fk+gh}}-1\right)\left[\frac{-fk}{gh}\right],$$

unde colligitur

INTEGRATIO CASUS IV

$$\int dz\sqrt{\frac{f+gzz}{-h+kzz}}$$

$$=C-\frac{fz}{h}\sqrt{\frac{-h+kzz}{f+gzz}}+\frac{f}{\sqrt{(fk+gh)}}\,\Pi\frac{fk+gh}{gh}\left(1-\frac{\sqrt{(fk+gh)}}{\sqrt{k(f+gzz)}}\right)\left[\frac{fk+gh}{gh}\right]$$

$$+\frac{fk+gh}{k\sqrt{fk}}\,\Pi\frac{fk}{gh}\left(\frac{\sqrt{k(f+gzz)}}{\sqrt{(fk+gh)}}-1\right)\left[\frac{-fk}{gh}\right].$$

COROLLARIUM 4

94. Si hic insuper g sumamus negative, prodit

INTEGRATIO CASUS VIII

$$\int dz \sqrt{\frac{f-gzz}{-h+kzz}}$$

$$= C - \frac{fz}{h}\sqrt{\frac{-h+kzz}{f-gzz}} + \frac{f}{\sqrt{(fk-gh)}}\, \Pi \frac{fk-gh}{gh}\left(\frac{\sqrt{(fk-gh)}}{\sqrt{k(f-gzz)}}-1\right)\left[\frac{-fk+gh}{gh}\right]$$
$$+ \frac{fk-gh}{k\sqrt{fk}}\, \Pi \frac{fk}{gh}\left(1-\frac{\sqrt{k(f-gzz)}}{\sqrt{(fk-gh)}}\right)\left[\frac{fk}{gh}\right]$$

hocque modo omnes plane 12 casus expedivimus.

CONCLUSIO

95. Intelligimus ergo duodecim casus formulae $\int dz \sqrt{\frac{f+gzz}{h+kzz}}$ supra enumeratos in tres classes distingui, quarum quaelibet quatuor casus complectatur. Prima scilicet classis eos continebit casus, quorum integratio simplici arcui sectionis conicae absolvitur, secunda vero eos, qui insuper partem algebraicam assumunt, at tertia classis praeter partem algebraicam duos arcus, alterum ellipticum, alterum hyperbolicum, postulat. Cum igitur in enumeratione casuum ad hunc ordinem non respexerimus, iam ita disponendi videntur.

Classis		Integralia exprimuntur
Prima	III, VI	arcu elliptico
	II, XII	arcu hyperbolico
Secunda	XI, IX	parte algebraica et arcu elliptico
	X, VII	parte algebraica et arcu hyperbolico
Tertia	I, IV, V, VIII	parte algebraica et duobus arcubus, altero elliptico, altero hyperbolico.

INTEGRATIO AEQUATIONIS

$$\frac{dx}{\sqrt{(A+Bx+Cx^2+Dx^3+Ex^4)}}=\frac{dy}{\sqrt{(A+By+Cy^2+Dy^3+Ey^4)}}$$

Commentatio 345 indicis Enestroemiani
Novi Commentarii academiae scientiarum Petropolitanae **12** (1766/7), 1768, p. 3—16
Summarium ibidem p. 5—6

SUMMARIUM

Calculus integralis, ad tantam hodie summorum Geometrarum studio perfectionem evectus, insignibus incrementis et subsidiis nunquam non ditatus fuit, quando ii aequationes differentiales solutu difficiliores, quarum integralia casu quasi vel per ambages et indirecte invenire ipsis licuerat, data opera meditationi subiecerunt methodos scrutaturi directas ad eadem, de quibus aliunde iam constitit, integralia perveniendi. Aequationis propositae integrale idque algebraicum et completum via admodum obliqua, cum in corporis ad duo centra virium fixa attracti motum inquireret, Ill. Eulero invenire licuit, qua is excitatus occasione istam integrationem data opera est aggressus eamque suis meditationibus eo censuit digniorem, quo plura et praeclariora Analyseos artificia difficultatum, quibus ea implicari videtur, evolutio, cum neutram partem seorsim ne ad arcus quidem circulares vel logarithmos revocare liceat, polliceri merito videbatur. En igitur directam methodum eamque substitutionibus et subsidiis analyticis notatu maxime dignis fundatam, qua propositae aequationis integrale eruitur cum priori perfecte congruens; quae cum sublatis difficultatibus potioribus dubium non sit, quin excoli possit uberius et ad brevitatem magis concinnam reduci, ad promovendos Analyseos fines plurimum momenti continere merito est censenda.

1. Methodo admodum singulari atque obliqua perveneram olim[1]) ad integrationem huius aequationis, cuius integrale idque adeo completum aequatione

1) L. Euleri Commentationes 251 et 261 (indicis Enestroemiani); vide p. 58 et 153. A. K.

algebraica inter x et y contineri deprehendi. Quod eo magis mirum videtur, quod utriusque formulae seorsim integrale non solum non algebraice, sed ne per circuli quidem hyperbolaeve quadraturam exprimi potest. Tum vero id imprimis notatu dignum occurrebat, quod nulla methodus directa patebat istud integrale algebraicum eruendi. Nulla autem occasio magis idonea videtur fines Analyseos proferendi, quam si, quod methodo obliqua quasi per ambages elicuerimus, idem methodo directa investigare annitamur. Cum igitur nuper[1] curvas definiverim, quas corpus ad duo centra virium fixa attractum percurrit, easque ad similem aequationem perduxerim, inde vicissim huius aequationis integrationem petere licebit; quod quomodo sit praestandum, hic explicare constitui.

2. Ac primo quidem observo aequationem propositam semper in eiusmodi formam transfundi posse, in qua coefficientes B et D evanescant, quod quidem de alterutro ex elementis satis est notum. Ut autem ambo simul ad nihilum redigi queant, id talis formae est proprium; posito enim $x=\frac{mz+a}{nz+b}$ prior forma, cui quidem altera est similis, abit in hanc

$$\frac{(mb-na)dz}{\sqrt{(A(nz+b)^4+B(nz+b)^3(mz+a)+C(nz+b)^2(mz+a)^2+D(nz+b)(mz+a)^3+E(mz+a)^4)}},$$

in cuius denominatore terminos tam ipsa quantitate z quam eius cubo z^3 affectos destruere licebit. Prior conditio praebet hanc aequationem

$$4Anb^3+Bmb^3+3Bnabb+2Cmabb+2Cnaab+3Dmaab+Dna^3+4Ema^3=0,$$

posterior vero hanc

$$4An^3b+Bn^3a+3Bmnnb+2Cmnna+2Cmmnb+3Dmmna+Dm^3b+4Em^3a=0,$$

unde tam ratio $a:b$ quam ratio $m:n$ elici potest.

1) L. EULERI Commentatio 301 (indicis ENESTROEMIANI): *De motu corporis ad duo centra virium fixa attracti*, Novi comment. acad. sc. Petrop. **10** (1764), 1766, p. 207; *LEONHARDI EULERI Opera omnia*, series II, vol. 5; Commentatio 328 (indicis ENESTROEMIANI): *De motu corporis ad duo centra virium fixa attracti*, Novi comment. acad. sc. Petrop. **11** (1765), 1767, p. 152; *LEONHARDI EULERI Opera omnia*, series II, vol. 5; Commentatio 337 (indicis ENESTROEMIANI): *Problème. Un corps étant attiré en raison réciproque quarrée des distances vers deux points fixes donnés, trouver les cas où la courbe décrite par ce corps sera algébrique*, Mém. de l'acad. d. sc. de Berlin **16** (1760), 1767, p. 228; *LEONHARDI EULERI Opera omnia*, series II, vol. 5. A. K.

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3. Ponamus enim $a = bp$ et $m = nq$, ut habeamus has aequationes

$$4A + Bq + 3Bp + 2Cpq + 2Cpp + 3Dppq + Dp^3 + 4Ep^3q = 0,$$
$$4A + Bp + 3Bq + 2Cpq + 2Cqq + 3Dpqq + Dq^3 + 4Epq^3 = 0,$$

quarum differentia per $p - q$ divisa praebet

$$2B + 2C(p+q) + D(pp + 4pq + qq) + 4Epq(p+q) = 0.$$

Tum vero prior per q [multiplicata] demta posteriore per p multiplicata dat divisione per $p - q$ facta

$$-4A - B(p+q) + Dpq(p+q) + 4Eppqq = 0;$$

statuamus nunc $p + q = r$ et $pq = s$ et ex aequationibus

$$2B + 2Cr + Drr + 2Ds + 4Ers = 0,$$
$$-4A - Br + Drs + 4Ess = 0$$

elidendo $r = \frac{4(A - Ess)}{Ds - B}$ adipiscimur hanc aequationem cubicam

$$\left.\begin{matrix} +D^3 \\ -4CDE \\ +8BEE \end{matrix}\right\} s^3 \left.\begin{matrix} -BDD \\ +4BCE \\ -8ADE \end{matrix}\right\} s^2 \left.\begin{matrix} -BBD \\ +4ACD \\ -8ABE \end{matrix}\right\} s \left.\begin{matrix} +B^3 \\ -4ABC \\ +8AAD \end{matrix}\right\} = 0,$$

unde incognita s definitur, quod igitur triplici modo fieri poterit.

4. Cum igitur sine detrimento scopi praefixi coefficientes B et D nihilo aequales assumere liceat, quaestio nostra in integrali huius aequationis inveniendo versatur

$$\frac{dx}{\sqrt{(A + Cxx + Dx^4)}} = \frac{dy}{\sqrt{(A + Cyy + Dy^4)}},$$

quam hoc modo repraesentemus

$$\frac{dx}{dy} = \sqrt{\frac{A + Cxx + Dx^4}{A + Cyy + Dy^4}},$$

unde relationem inter variabiles x et y generatim elici oportet, id quod sequenti modo praestare conabor.

5. Ponamus primo $x=\sqrt{npq}$ et $y=\sqrt{n\frac{p}{q}}$; erit

$$dx=\frac{\sqrt{n}(qdp+pdq)}{2\sqrt{pq}} \quad \text{et} \quad dy=\frac{\sqrt{n}(qdp-pdq)}{2q\sqrt{pq}}$$

hincque

$$\frac{dx}{dy}=\frac{q(qdp+pdq)}{qdp-pdq}.$$

Porro autem est

$$\frac{A+Cxx+Dx^4}{A+Cyy+Dy^4}=\frac{qq(A+nCpq+nnDppqq)}{Aqq+nCpq+nnDpp},$$

unde fit

$$\frac{qdp+pdq}{qdp-pdq}=\sqrt{\frac{A+nCpq+nnDppqq}{Aqq+nCpq+nnDpp}},$$

ubi nunc numerus n ad commodum nostrum assumi potest.

6. Sit brevitatis gratia

$$\frac{A+nCpq+nnDppqq}{Aqq+nCpq+nnDpp}=\frac{P+Q}{P-Q};$$

erit

$$\frac{P}{Q}=\frac{A(1+qq)+2nCpq+nnDpp(1+qq)}{A(1-qq)-nnDpp(1-qq)}=\frac{(A+nnDpp)(1+qq)+2nCpq}{(A-nnDpp)(1-qq)}.$$

Tum vero ob

$$\frac{qdp+pdq}{qdp-pdq}=\sqrt{\frac{P+Q}{P-Q}}$$

obtinebimus

$$\frac{qdp}{pdq}=\frac{\sqrt{(P+Q)}+\sqrt{(P-Q)}}{\sqrt{(P+Q)}-\sqrt{(P-Q)}}=\frac{P+\sqrt{(PP-QQ)}}{Q}$$

et

$$\frac{pdq}{qdp}=\frac{P-\sqrt{(PP-QQ)}}{Q}.$$

7. Omne iam momentum versatur in idonea substitutione; atque equidem hac utendum observavi

$$q=u+\sqrt{(uu-1)}, \quad \text{unde fit} \quad \frac{dq}{q}=\frac{du}{\sqrt{(uu-1)}}$$

et porro

$$1+qq=2qu, \quad 1-qq=-2q\sqrt{(uu-1)},$$

ex quo conficitur

$$\frac{P}{Q}=\frac{(A+nnDpp)u+nCp}{(nnDpp-A)\sqrt{(uu-1)}},$$

ac nunc quidem pro n unitatem commodissime assumi evidens est. Cum ergo sit

$$\frac{P}{Q}=\frac{(A+Dpp)u+Cp}{(Dpp-A)\sqrt{(uu-1)}},$$

erit

$$\frac{\sqrt{(PP-QQ)}}{Q}=\frac{\sqrt{(4ADppuu+2Cp(A+Dpp)u+CCpp+(Dpp-A)^2)}}{(Dpp-A)\sqrt{(uu-1)}},$$

ita ut nostra aequatio integranda sit

$$\frac{pdu}{dp}=\frac{(A+Dpp)u+Cp-\sqrt{(4ADppuu+2Cpu(A+Dpp)+CCpp+(Dpp-A)^2)}}{Dpp-A}.$$

8. Ista formula irrationalis hoc modo repraesentetur

$$\sqrt{\left(\left(2pu\sqrt{AD}+\frac{C(A+Dpp)}{2\sqrt{AD}}\right)^2+\frac{(4AD-CC)(Dpp-A)^2}{4AD}\right)}$$

ac ponatur

$$2pu\sqrt{AD}+\frac{C(A+Dpp)}{2\sqrt{AD}}=\frac{(Dpp-A)s\sqrt{(4AD-CC)}}{2\sqrt{AD}},$$

unde fit ipsa formula surda

$$=\frac{(Dpp-A)\sqrt{(4AD-CC)(1+ss)}}{2\sqrt{AD}}$$

et

$$u=-\frac{C(A+Dpp)}{4ADp}+\frac{(Dpp-A)s\sqrt{(4AD-CC)}}{4ADp}$$

hincque

$$(A+Dpp)u+Cp=\frac{-C(Dpp-A)^2+(A+Dpp)(Dpp-A)s\sqrt{(4AD-CC)}}{4ADp},$$

ita ut iam nostra aequatio sit

$$\frac{pdu}{dp}=\frac{-C(Dpp-A)+(A+Dpp)s\sqrt{(4AD-CC)}}{4ADp}-\frac{\sqrt{(4AD-CC)(1+ss)}}{2\sqrt{AD}}.$$

9. Inde vero colligimus

$$du=\frac{-Cdp(Dpp-A)}{4ADpp}+\frac{sdp(A+Dpp)\sqrt{(4AD-CC)}}{4ADpp}+\frac{ds(Dpp-A)\sqrt{(4AD-CC)}}{4ADp},$$

ita ut obtineamus

$$\frac{pdu}{dp}=\frac{-C(Dpp-A)}{4ADp}+\frac{s(A+Dpp)\sqrt{(4AD-CC)}}{4ADp}+\frac{ds(Dpp-A)\sqrt{(4AD-CC)}}{4ADdp},$$

qua formula praecedenti aequata commodissime usu venit, ut plerique termini sponte se tollant indeque exsurgat haec aequatio

$$\frac{ds(Dpp-A)\sqrt{(4AD-CC)}}{4ADdp}=\frac{-\sqrt{(4AD-CC)}(1+ss)}{2\sqrt{AD}},$$

unde nascitur

$$\frac{ds}{\sqrt{(1+ss)}}=\frac{-2dp\sqrt{AD}}{Dpp-A}=\frac{2dp\sqrt{AD}}{A-Dpp},$$

cuius integrale in logarithmis est

$$l\left(s+\sqrt{(1+ss)}\right)=l\frac{\sqrt{A}+p\sqrt{D}}{\sqrt{A}-p\sqrt{D}}+l\alpha,$$

ita ut habeamus

$$s+\sqrt{(1+ss)}=\frac{\alpha\sqrt{A}+\alpha p\sqrt{D}}{\sqrt{A}-p\sqrt{D}}$$

hincque

$$s=\frac{\alpha\alpha(\sqrt{A}+p\sqrt{D})^2-(\sqrt{A}-p\sqrt{D})^2}{2\alpha(A-Dpp)}.$$

10. Quodsi hinc regrediamur, reperiemus

$$u=\frac{-C(A+Dpp)}{4ADp}+\frac{(\sqrt{A}-p\sqrt{D})^2-\alpha\alpha(\sqrt{A}+p\sqrt{D})^2}{8\alpha ADp}\sqrt{(4AD-CC)},$$

unde definiri oportet $q=u+\sqrt{(uu-1)}$. Sed quia hinc fit $u=\frac{1+qq}{2q}$, restituendo $p=xy$ et $q=\frac{x}{y}$ aequatio nostra integralis completa est

$$\frac{xx+yy}{2xy}=\frac{-C(A+Dxxyy)}{4ADxy}+\frac{(\sqrt{A}-xy\sqrt{D})^2-\alpha\alpha(\sqrt{A}+xy\sqrt{D})^2}{8\alpha ADxy}\sqrt{(4AD-CC)}$$

seu

$$4AD(xx+yy)+2C(A+Dxxyy)$$
$$=\frac{\sqrt{(4AD-CC)}}{\alpha}\left(\left(\sqrt{A}-xy\sqrt{D}\right)^2-\alpha\alpha\left(\sqrt{A}+xy\sqrt{D}\right)^2\right),$$

quae evolvitur in hanc

$$\frac{4AD(xx+yy)+2C(A+Dxxyy)}{\sqrt{(4AD-CC)}}=\frac{(1-\alpha\alpha)A-2(1+\alpha\alpha)xy\sqrt{AD}+(1-\alpha\alpha)Dxxyy}{\alpha},$$

et ponendo

$$\alpha=\frac{\sqrt{(4AD-CC)}}{mC}$$

prodit

$$4AD(xx+yy)+2C(A+Dxxyy)$$
$$=\frac{((1+mm)CC-4AD)(A+Dxxyy)-2((mm-1)CC+4AD)xy\sqrt{AD}}{mC}.$$

11. Ne casus, ubi $\sqrt{AD}$ fit quantitas imaginaria, turbent, iuvabit integrationem alia via, quae ipsa destructione terminorum § 9 observata innitatur, investigare. Scilicet proposita aequatione

$$\frac{dx}{dy}=\sqrt{\frac{A+Cxx+Ex^4}{A+Cyy+Ey^4}}$$

fiat $x=\sqrt{pq}$ et $y=\sqrt{\frac{p}{q}}$, ut hinc obtineatur

$$\frac{pdq}{qdp}=\frac{P-\sqrt{(PP-QQ)}}{Q}$$

existente

$$\frac{P}{Q}=\frac{(A+Epp)(1+qq)+2Cpq}{(A-Epp)(1-qq)}.$$

Ponatur nunc $q=u+\sqrt{(uu-1)}$, ut sit

$$1+qq=2qu,\quad 1-qq=2qu-2qq=-2q\sqrt{(uu-1)};$$

erit

$$\frac{dq}{q}=\frac{du}{\sqrt{(uu-1)}}\quad\text{et}\quad\frac{P}{Q}=\frac{u(A+Epp)+Cp}{(Epp-A)\sqrt{(uu-1)}},$$

unde resultat haec aequatio transformata

$$\frac{pdu}{dp}=\frac{u(A+Epp)+Cp-\sqrt{(4AEppuu+2Cpu(A+Epp)+CCpp+(Epp-A)^2)}}{Epp-A}.$$

12. Hac aequatione in ordinem redacta et posito brevitatis gratia membro irrationali $=\sqrt{M}$ fiet

$$udp(A+Epp)+Cpdp-pdu(Epp-A)=dp\sqrt{M}$$

ac reiecto primum hoc membro irrationali reperitur integrale

$$\frac{C+2Epu}{Epp-A}=\text{Const.};$$

cuius constantis loco autem sumatur quantitas variabilis s, ut sit

$$2Epu+C=s(Epp-A) \quad \text{et} \quad u=\frac{s(Epp-A)-C}{2Ep},$$

atque hinc membrum rationale fit

$$\frac{-ds(Epp-A)^2}{2E}$$

et formula irrationalis

$$(Epp-A)\sqrt{\frac{Ass+Cs+E}{E}},$$

ita ut nunc sit

$$\frac{ds}{2}(Epp-A)=dp\sqrt{E(Ass+Cs+E)}$$

seu

$$\frac{ds}{\sqrt{E(Ass+Cs+E)}}+\frac{2dp}{Epp-A}=0,$$

cuius integrale est

$$\frac{1}{\sqrt{AE}}l\frac{p\sqrt{E}-\sqrt{A}}{p\sqrt{E}+\sqrt{A}}+\frac{1}{\sqrt{AE}}l\left(As+\frac{1}{2}C+\sqrt{A(Ass+Cs+E)}\right)=\text{Const.}$$

13. Haec aequatio ergo redit ad hanc formam

$$As+\frac{1}{2}C+\sqrt{A(Ass+Cs+E)}=\alpha\frac{p\sqrt{E}+\sqrt{A}}{p\sqrt{E}-\sqrt{A}}=T,$$

unde elicitur

$$AE=TT-T(2As+C)+\tfrac{1}{4}CC$$

seu

$$2As+C=\frac{TT+\frac{1}{4}CC-AE}{T}=\frac{\alpha\alpha(p\sqrt{E}+\sqrt{A})^2+(\frac{1}{4}CC-AE)(p\sqrt{E}-\sqrt{A})^2}{\alpha(Epp-A)}.$$

Cum nunc sit $p = xy$ et $q = \frac{x}{y}$, erit

$$u = \frac{xx+yy}{2xy} \quad \text{et} \quad s = \frac{E(xx+yy)+C}{Exxyy-A},$$

ex quo efficitur

$$\frac{2AE(xx+yy)+CExxyy+AC}{Exxyy-A} = T + \frac{CC-4AE}{4T}$$

existente

$$T = \alpha\cdot\frac{xy\sqrt{E}+\sqrt{A}}{xy\sqrt{E}-\sqrt{A}} = \alpha\cdot\frac{Exxyy+A+2xy\sqrt{AE}}{Exxyy-A}$$

et

$$\frac{1}{T} = \frac{1}{\alpha}\cdot\frac{Exxyy+A-2xy\sqrt{AE}}{Exxyy-A}$$

ideoque

$$2AE(xx+yy)+CExxyy+AC = \alpha(Exxyy+A)+2\alpha xy\sqrt{AE} + \frac{CC-4AE}{4\alpha}(Exxyy+A) - \frac{2(CC-4AE)}{4\alpha}xy\sqrt{AE}.$$

14. Ne unquam haec expressio involvat imaginaria, constantis α formam ita immutemus, ut sit

$$\alpha + \frac{CC-4AE}{4\alpha} = F \quad \text{seu} \quad 4\alpha\alpha = 4\alpha F - CC + 4AE$$

hincque

$$2\alpha = F + \sqrt{(FF+4AE-CC)} \quad \text{et} \quad \frac{1}{2\alpha} = \frac{F-\sqrt{(FF+4AE-CC)}}{CC-4AE},$$

unde fit

$$2\alpha - \frac{CC-4AE}{2\alpha} = 2\sqrt{(FF+4AE-CC)}$$

et

$$2AE(xx+yy) = (F-C)(Exxyy+A)+2xy\sqrt{AE(FF+4AE-CC)};$$

sit nunc $F - C = 2G$; erit

$$AE(xx+yy) = G(A+Exxyy)+2xy\sqrt{AE(AE+CG+GG)},$$

quae est aequatio integralis completa huius differentialis

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} = \frac{dy}{\sqrt{(A+Cyy+Ey^4)}},$$

ubi constans G ita accipi debet, ut formula irrationalis

$$\sqrt{AE(AE+CG+GG)}$$

non fiat imaginaria.

15. Forma haec integralis adhuc commodior reddi potest ponendo $G=Eff$ sicque fiet aequatio integralis

$$A(xx+yy)=ff(A+Exxyy)+2xy\sqrt{A(A+Cff+Ef^4)},$$

ubi f est constans arbitraria. Hinc autem elicitur

$$y=\frac{x\sqrt{A(A+Cff+Ef^4)}\pm f\sqrt{A(A+Cxx+Ex^4)}}{A-Effxx}$$

similique modo

$$x=\frac{y\sqrt{A(A+Cff+Ef^4)}\pm f\sqrt{A(A+Cyy+Ey^4)}}{A-Effyy}.$$

Quae formulae cum iis, quas olim[1] dederam, perfecte consentiunt.

16. Integrale hic quidem aequationis differentialis propositae methodo directa sum consecutus, verumtamen diffiteri non possum hoc per multas ambages esse praestitum, ita ut vix sit expectandum cuiquam has operationes in mentem venire potuisse. Ex quo haec ipsa methodus, qua hic sum usus, plurimum in recessu habere videtur neque ullum est dubium, quin eam diligentius scrutando aditus ad multa alia praeclara aperiatur ac fortasse alia nova methodus idem praestandi detegatur, unde non contemnenda subsidia ad Analysin perficiendam hauriri queant.

17. Operationes hic adhibitae aliquantum variari possunt, quod probe perpendisse usu non carebit. Propositam scilicet aequationem differentialem ita refero

$$\frac{ydx}{xdy}=\sqrt{\frac{Ayy+Cxxyy+Ex^4yy}{Axx+Cxxyy+Exxy^4}}=\sqrt{\frac{P+Q}{P-Q}},$$

ut sit

$$\frac{P}{Q}=\sqrt{\frac{(A+Exxyy)(xx+yy)+2Cxxyy}{(A-Exxyy)(yy-xx)}},$$

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1) L. EULERI Commentatio 261 (indicis ENESTROEMIANI); vide p. 153. A. K.

eritque

$$\frac{ydx+xdy}{ydx-xdy}=\frac{\sqrt{(P+Q)}+\sqrt{(P-Q)}}{\sqrt{(P+Q)}-\sqrt{(P-Q)}}=\frac{P+\sqrt{(PP-QQ)}}{Q},$$

tum etiam

$$\frac{ydx-xdy}{ydx+xdy}=\frac{P-\sqrt{(PP-QQ)}}{Q}.$$

Faciamus nunc hanc substitutionem

$$x=p\left(\sqrt{\frac{q+1}{2}}-\sqrt{\frac{q-1}{2}}\right) \quad \text{et} \quad y=p\left(\sqrt{\frac{q+1}{2}}+\sqrt{\frac{q-1}{2}}\right);$$

erit

$$xy=pp, \quad xx+yy=2ppq, \quad yy-xx=2pp\sqrt{(qq-1)},$$

deinde

$$\frac{dx}{x}=\frac{dp}{p}-\frac{dq}{2\sqrt{(qq-1)}} \quad \text{et} \quad \frac{dy}{y}=\frac{dp}{p}+\frac{dq}{2\sqrt{(qq-1)}};$$

unde fit

$$\frac{ydx}{xdy}=\frac{\frac{dp}{p}-\frac{dq}{2\sqrt{(qq-1)}}}{\frac{dp}{p}+\frac{dq}{2\sqrt{(qq-1)}}} \quad \text{et} \quad \frac{ydx-xdy}{ydx+xdy}=\frac{-pdq}{2dp\sqrt{(qq-1)}}$$

atque

$$\frac{P}{Q}=\frac{2(A+Ep^4)ppq+2Cp^4}{2(A-Ep^4)pp\sqrt{(qq-1)}}=\frac{(A+Ep^4)q+Cpp}{(A-Ep^4)\sqrt{(qq-1)}},$$

unde fit

$$\frac{\sqrt{(PP-QQ)}}{Q}=\frac{\sqrt{(4AEp^4qq+2Cppq(A+Ep^4)+CCp^4+(A-Ep^4)^2)}}{(A-Ep^4)\sqrt{(qq-1)}}.$$

18. Sit $pp=r$ eritque ob $\frac{dp}{p}=\frac{dr}{2r}$

$$0=\frac{rdq}{dr}+\frac{(A+Err)q+Cr-\sqrt{(4AErrqq+2Crq(A+Err)+CCrr+(A-Err)^2)}}{A-Err}$$

sive

$$\begin{aligned}&rdq(A-Err)+qdr(A+Err)+Crdr\\&=dr\sqrt{\left(4AErrqq+2Crq(A+Err)+CCrr+(A-Err)^2\right)}.\end{aligned}$$

Quantitas vinculo radicali implicata ita exhibeatur

$$\begin{aligned}&\frac{1}{4AE}(16AAEErrqq+8ACErq(A+Err)+4ACCErr+4AE(A-Err)^2)\\&=\frac{1}{4AE}\left((4AErq+C(A+Err))^2+(4AE-CC)(A-Err)^2\right).\end{aligned}$$

Ponamus ergo

$$4AErq + C(A+Err) = s(A-Err)\sqrt{(4AE-CC)}$$

eritque formula surda

$$= \frac{(A-Err)\sqrt{(4AE-CC)}(1+ss)}{2\sqrt{AE}}$$

et ob

$$s\sqrt{(4AE-CC)} = \frac{4AErq + C(A+Err)}{A-Err}$$

erit differentiando

$$ds\sqrt{(4AE-CC)} = \frac{4AAE(rdq+qdr) - 4AEEr^3dq + 4AEErrqdr + 4ACErdr}{(A-Err)^2}$$

ideoque

$$rdq(A-Err) + qdr(A+Err) + Crdr = \frac{ds(A-Err)^2\sqrt{(4AE-CC)}}{4AE};$$

quod cum sit ipsum prius membrum nostrae aequationis, cui aequalis est

$$\frac{dr(A-Err)\sqrt{(4AE-CC)}(1+ss)}{2\sqrt{AE}},$$

habebimus

$$\frac{ds(A-Err)}{2\sqrt{AE}} = dr\sqrt{(1+ss)} \quad \text{et} \quad \frac{2dr\sqrt{AE}}{A-Err} = \frac{ds}{\sqrt{(1+ss)}},$$

cuius integrale est

$$s + \sqrt{(1+ss)} = \alpha \cdot \frac{\sqrt{A}+r\sqrt{E}}{\sqrt{A}-r\sqrt{E}},$$

unde fit

$$1 = \alpha\alpha\left(\frac{\sqrt{A}+r\sqrt{E}}{\sqrt{A}-r\sqrt{E}}\right)^2 - 2\alpha s\cdot\frac{\sqrt{A}+r\sqrt{E}}{\sqrt{A}-r\sqrt{E}}.$$

Est vero

$$s = \frac{4AEqr + C(A+Err)}{(A-Err)\sqrt{(4AE-CC)}}$$

atque

$$r = pp = xy \quad \text{et} \quad q = \frac{xx+yy}{2xy}$$

hincque

$$s = \frac{2AE(xx+yy) + C(A+Exxyy)}{(A-Exxyy)\sqrt{(4AE-CC)}}.$$

19. Idem expedire possumus sine substitutione nova; statim enim ac pervenimus ad hanc aequationem

$$rdq(A-Err)+qdr(A+Err)+Crdr$$
$$=dr\sqrt{\frac{(4AErq+C(A+Err))^2+(4AE-CC)(A-Err)^2}{4AE}},$$

notetur esse membrum prius

$$=\frac{(A-Err)^2}{4AE}d.\frac{4AErq+C(A+Err)}{A-Err},$$

posterius vero ita exprimi posse

$$\frac{dr(A-Err)}{2\sqrt{AE}}\sqrt{\left(4AE-CC+\left(\frac{4AErq+C(A+Err)}{A-Err}\right)^2\right)};$$

unde posito brevitatis gratia

$$\frac{4AErq+C(A+Err)}{A-Err}=v$$

erit

$$\frac{(A-Err)^2}{4AE}dv=\frac{dr(A-Err)}{2\sqrt{AE}}\sqrt{(4AE-CC+vv)}$$

ideoque

$$\frac{dv}{\sqrt{(4AE-CC+vv)}}=\frac{2dr\sqrt{AE}}{A-Err}.$$

20. Aliud specimen huius reductionis daturus considerabo hanc aequationem

$$\frac{dx}{\sqrt{(Bx+Cxx+Dx^3)}}=\frac{dy}{\sqrt{(By+Cyy+Dy^3)}},$$

quam ita repraesento

$$\frac{ydx}{xdy}=\sqrt{\frac{Bxyy+Cxxyy+Dx^3yy}{Bxxy+Cxxyy+Dxxy^3}}=\sqrt{\frac{P+Q}{P-Q}},$$

ut sit

$$\frac{P}{Q}=\frac{Bxy(y+x)+2Cxxyy+Dxxyy(x+y)}{Bxy(y-x)+Dxxyy(x-y)}$$

seu

$$\frac{P}{Q}=\frac{(B+Dxy)(x+y)+2Cxy}{(B-Dxy)(y-x)},$$

eritque

$$\frac{ydx-xdy}{ydx+xdy}=\frac{P+\sqrt{(PP-QQ)}}{Q}.$$

21. Statuatur nunc

$$x=p\left(u+\sqrt{(uu-1)}\right) \quad \text{et} \quad y=p\left(u-\sqrt{(uu-1)}\right);$$

erit

$$\frac{dx}{x}=\frac{dp}{p}+\frac{du}{\sqrt{(uu-1)}} \quad \text{et} \quad \frac{dy}{y}=\frac{dp}{p}-\frac{du}{\sqrt{(uu-1)}}$$

hincque

$$\frac{ydx-xdy}{ydx+xdy}=\frac{pdu}{dp\sqrt{(uu-1)}}.$$

Deinde ob

$$xy=pp \quad \text{et} \quad x+y=2pu, \quad y-x=-2p\sqrt{(uu-1)}$$

erit

$$\frac{P}{Q}=\frac{(B+Dpp)u+Cp}{-(B-Dpp)\sqrt{(uu-1)}}$$

ideoque

$$\frac{pdu}{dp}=\frac{(B+Dpp)u+Cp-\sqrt{(4BDppuu+2Cpu(B+Dpp)+CCpp+(B-Dpp)^2)}}{Dpp-B},$$

unde fit

$$udp(B+Dpp)-pdu(Dpp-B)+Cpdp=dp\sqrt{(\ldots.)}.$$

Prius membrum est

$$(B-Dpp)^2 d.\frac{pu+\frac{C}{4BD}(B+Dpp)}{B-Dpp}$$

seu

$$\frac{(B-Dpp)^2}{4BD}\,d.\frac{4BDpu+C(B+Dpp)}{B-Dpp},$$

at quantitas signo radicali involuta ita scribi potest

$$\frac{1}{4BD}\left(16BBDDppuu+8BCDpu(B+Dpp)+4BCCDpp+4BD(B-Dpp)^2\right)$$

$$=\frac{1}{4BD}\left((4BDpu+C(B+Dpp))^2+(4BD-CC)(B-Dpp)^2\right),$$

unde membrum irrationale erit

$$\frac{B-Dpp}{2\sqrt{BD}}\sqrt{\left(4BD-CC+\left(\frac{4BDpu+C(B+Dpp)}{B-Dpp}\right)^2\right)}.$$

Quare posito brevitatis gratia

$$\frac{4BDpu+C(B+Dpp)}{B-Dpp}=s$$

erit

$$\frac{(B-Dpp)^2}{4BD}ds=\frac{(B-Dpp)dp}{2\sqrt{BD}}\sqrt{(4BD-CC+ss)},$$

unde fit

$$\frac{ds}{\sqrt{(4BD-CC+ss)}}=\frac{2dp\sqrt{BD}}{B-Dpp}$$

et integrando

$$s+\sqrt{(4BD-CC+ss)}=\alpha\cdot\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}$$

ideoque

$$4BD-CC=\alpha\alpha\left(\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}\right)^2-2\alpha s\cdot\frac{\sqrt{B}+p\sqrt{D}}{\sqrt{B}-p\sqrt{D}}.$$

22. Fundamentum ergo harum reductionum in hoc consistit, ut primo ponatur $x=pq$ et $y=\frac{p}{q}$, tum vero pro q eiusmodi formula accipiatur, qua partes $x\pm y$, $xx\pm yy$ etc., quae in formula $\frac{P}{Q}$ insunt, quam simplicissimae reddantur. Veluti in casu § 17 sumsimus

$$q=\sqrt{\frac{u+1}{2}}+\sqrt{\frac{u-1}{2}}$$

seu $qq=u+\sqrt{(uu-1)}$, in ultimo vero $q=u+\sqrt{(uu-1)}$; ibi nempe opus non erat, ut $x+y$ rationaliter exprimatur, unde sufficiebat ipsi qq formam $u+\sqrt{(uu-1)}$ tribui, hic vero necesse erat, ut $x+y$ rationalem consequatur valorem.

23. Denique casum simpliciorem praetermittere non possum, quo proponitur haec aequatio

$$\frac{dx}{\sqrt{(A+Cxx)}}=\frac{dy}{\sqrt{(A+Cyy)}},$$

quam ita refero

$$\frac{ydx}{xdy}=\sqrt{\frac{Ayy+Cxxyy}{Axx+Cxxyy}}=\sqrt{\frac{P+Q}{P-Q}};$$

posito ergo

$$x=p\left(\sqrt{\frac{q+1}{2}}-\sqrt{\frac{q-1}{2}}\right)\quad\text{et}\quad y=p\left(\sqrt{\frac{q+1}{2}}+\sqrt{\frac{q-1}{2}}\right)$$

fiet

$$\frac{-pdq}{2dp\sqrt{(qq-1)}}=\frac{P-\sqrt{(PP-QQ)}}{Q}$$

existente

$$\frac{P}{Q}=\frac{Aq+Cpp}{A\sqrt{(qq-1)}} \quad \text{et} \quad \frac{\sqrt{(PP-QQ)}}{Q}=\frac{\sqrt{(2ACppq+CCp^4+AA)}}{A\sqrt{(qq-1)}},$$

unde sumto $pp=r=xy$ erit

$$0=\frac{rdq}{dr}+\frac{Aq+Cr-\sqrt{(2ACrq+CCrr+AA)}}{A}$$

hincque

$$\frac{A(rdq+qdr)+Crdr}{\sqrt{(2ACrq+CCrr+AA)}}=dr,$$

cuius integrale est

$$Cr+F=\sqrt{(2ACrq+CCrr+AA)} \quad \text{seu} \quad FF+2CFr=2ACrq+AA;$$

est vero

$$r=xy \quad \text{et} \quad q=\frac{xx+yy}{2xy},$$

unde aequatio integralis est

$$FF+2CFxy=AA+AC(xx+yy).$$

Sicque haec comparatio inter x et y, quae alias per logarithmos vel arcus circulares ostendi solet, hic algebraice est eruta.

EVOLUTIO GENERALIOR FORMULARUM COMPARATIONI CURVARUM INSERVIENTIUM

Commentatio 347 indicis ENESTROEMIANI
Novi commentarii acad. sc. Petrop. **12** (1766/7), 1768, p. 42—86
Summarium ibidem p. 9—10

SUMMARIUM

Insignia sunt et miro cum ingenii acumine excogitata, quae Ill. Comes FAGNANUS de comparatione arcuum curvae lemniscatae elicuit quaeque non minori sagacitate circa arcus ellipticos atque etiam hyperbolicos inter se comparandos est commentatus. Methodum illius Geometrarum attentione dignissimam iam pridem in hisce Commentariis Ill. EULERUS suis meditationibus non illustravit modo, sed longe etiam reddidit generaliorem methodum exponendo planam a substitutionibus admodum molestis, quibus FAGNANUS usus est et quarum ratio inventionis prorsus est obscura, liberam atque generalissime omnes istorum arcuum comparationes in se complexam, cuius ideo beneficio ipsi in gravissimo hoc negotio multo longius progredi licuit. Ad duo vero potissimum capita arduam sane hanc quaestionem revocare licet, dum scilicet demonstravit Cel. EULERUS primo quidem omnium curvarum, quarum rectificatio hac integrali formula contineatur

$$\int \frac{\mathfrak{A}\,dz}{\sqrt{(A + Cz^2 + Ez^4)}},$$

arcus perinde atque circulares inter se comparari posse, ita ut sumto in istis curvis arcu quocunque ab alio quovis puncto arcus geometrice abscindi possit, qui ad illum rationem quamcunque rationalem teneat; deinde vero in curvis, quarum rectificatio ab ista formula

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z^2 + \mathfrak{C}z^4 + \mathfrak{D}z^6 + \text{etc.})}{\sqrt{(A + Cz^2 + Ez^4)}}$$

pendeat, omnia ea aeque felici successu expediri, quae iam pridem circa comparationem arcuum parabolicorum praeclara sunt inventa, ita ut in modo memoratis curvis sumto arcu quocunque ab alio quovis puncto arcus abscindi possit, qui ab illo vel a quovis eius multiplo quantitate differat vel geometrice assignabili vel a circuli hyperbolaeve quadratura pendente.

Insigne vero Ill. Auctor profundissimae huic investigationi incrementum attulit methodum suam ad istas quoque formulas extendendo, qui expressionem surdam magis complicatam

$$\sqrt{(A+2Bz+Cz^2+2Dz^3+Ez^4)}$$

involvunt; quo ipso latissimus aperitur campus in aliis pluribus curvis similes comparationes instituendi. Quod argumentum cum non ad curvarum modo naturam profundius scrutandam summum praestet usum, sed largissimam quoque gravissimarum ad Analysin perficiendam observationum messem sistat, in praesenti dissertatione plene evolvitur; cui si addantur ea, quae Ill. EULERUS in *Calculi* sui *integralis* typis in Academia nostra exscripti Vol. I Sect. II Cap. V et VI de istis formulis integralibus est commentatus, gravissimam quaestionem ad insigne Analyseos incrementum in plena luce positam esse est, quod laetentur Geometrae.

1. Quae de comparatione arcuum circularium ex elementis sunt cognita et quae Illustr. Comes FAGNANUS de simili comparatione arcuum curvae lemniscatae mira sagacitate elicuit, ea, uti iam aliquoties[1]) ostendi, ita generalius enunciari possunt, ut, si cuiuspiam lineae curvae arcus indefinite per hanc formulam integralem exprimatur

$$\int\frac{\mathfrak{A}dz}{\sqrt{(A+Czz+Ez^4)}},$$

tum in ea curva sumto arcu quocunque ab alio quovis puncto arcum geometrice abscindi posse illi arcui aequalem. Atque hinc etiam proposito arcu quocunque ab alio quovis puncto arcus abscindi poterit, qui illius arcus sit duplus seu triplus seu qui in genere ad eum rationem quamcunque rationalem teneat. Unde consequitur omnium curvarum, quarum quidem rectificatio ista formula contineatur, arcus perinde atque arcus circulares inter se comparari posse.

1) L. EULERI Commentationes 251, 261, 264 (indicis ENESTROEMIANI); vide p. 58, 153, 201.
A. K.

2. Deinde quae de comparatione arcuum parabolicorum iam pridem sunt inventa et quae simili modo Ill. Comes FAGNANUS circa arcus ellipticos et hyperbolicos summo acumine praestitit, ea deinceps tam late patere demonstravi, ut pari successu ad omnes curvas, quarum arcus indefinite per hanc formulam integralem

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}zz + \mathfrak{C}z^4 + \mathfrak{D}z^6 + \text{etc.})}{\sqrt{(A + Czz + Dz^4)}}$$

exprimatur, extendi queant. Sumto scilicet in tali curva arcu quocunque ab alio quovis puncto arcus abscindi poterit, qui ab illo arcu differat quantitate geometrice assignabili. Tum vero etiam abscindi poterunt eiusmodi arcus, qui ab arcus propositi duplo, triplo vel quovis multiplo differant quantitate geometrice assignabili. Quin etiam illud punctum, unde arcus abscindi oportet, ita capi poterit, ut haec differentia plane in nihilum abeat.

3. Quaecunque ergo circa arcus parabolicos iam olim sunt praestita, eadem quoque in omnibus curvis, quarum rectificatio ad istam formulam integralem est reductibilis, pari successu expediri poterunt. Cum autem Comes FAGNANUS ad has mirabiles comparationes per substitutiones admodum molestas, et quarum ratio inventionis ne quidem perspiciatur, pervenerit, ego methodum planam aperui, quae quasi sponte ad easdem comparationes manuducat. Atque ista methodus etiam multo uberius hoc negotium conficit, quod generalissime omnes comparationes in se complectitur; aequivalet enim integrationi completae, quae simul constantem arbitrariam involvit, dum illae substitutiones tantum integrationes particulares referre sunt censendae, quam ob causam mihi quidem huius methodi beneficio multo longius progredi licuit, uti ex aliquot speciminibus, quae iam dedi, luculenter apparet.

4. Quemadmodum autem in his formulis, quas pertractavi, ista expressio surda $\sqrt{(A + Czz + Ez^4)}$ implicatur, quae quidem iam casus solutu difficillimos complectitur, ita eadem ad expressionem surdam magis complicatam hanc

$$\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}$$

extendi posse observavi; qua multo amplior campus aperitur similes comparationes in pluribus aliis lineis curvis instituendi. Neque vero haec investigatio tantum in lineis curvis tam eximium praestat usum, sed etiam in Ana-

lysi et Calculo integrali gravissima incrementa largiri videtur; ad quae plenius excolenda ut viam sternam, evolutiones ad hanc formulam generaliorem pertinentes diligentius exponam. Hunc in finem proposita sit sequens aequatio relationem inter binas variabiles x et y exprimens.

AEQUATIO CANONICA EXPENDENDA

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

5. Haec aequatio praeter binas variabiles x et y continet sex quantitates constantes, quae autem, cum tantum earum ratio spectetur, ad quinque reducuntur, ita ut quinque determinationes ab arbitrio nostro pendentes recipere sit censendum. Deinde etsi haec aequatio ratione variabilium ad quatuor dimensiones exsurgit, tamen utraque seorsim nusquam ultra duas ascendit, ita ut utriusque valor per resolutionem aequationis quadraticae exhiberi queat, id quod praesens institutum necessario postulat. Denique ambae variabiles x et y in hanc aequationem aequaliter ingrediuntur, et etiamsi permutentur, nullam mutationem inducunt, ut utraque per alteram formula omnino simili exprimatur. Atque ob has rationes membra x^3+y^3, x^4+y^4 et $xy(xx+yy)$ uti et altiores dimensiones omitti oportuit.

6. Quodsi iam ex hac aequatione tam valorem ipsius x quam ipsius y extrahamus, reperiemus

$$x = \frac{-\beta-\delta y-\varepsilon yy \pm \sqrt{((\beta+\delta y+\varepsilon yy)^2-(\alpha+2\beta y+\gamma yy)(\gamma+2\varepsilon y+\zeta yy))}}{\gamma+2\varepsilon y+\zeta yy},$$

$$y = \frac{-\beta-\delta x-\varepsilon xx \pm \sqrt{((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx))}}{\gamma+2\varepsilon x+\zeta xx}.$$

Ponamus brevitatis gratia

$$\pm\sqrt{\big((\beta+\delta y+\varepsilon yy)^2-(\alpha+2\beta y+\gamma yy)(\gamma+2\varepsilon y+\zeta yy)\big)} = Y,$$

$$\pm\sqrt{\big((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx)\big)} = X,$$

ut habeamus

$$x = \frac{-\beta-\delta y-\varepsilon yy+Y}{\gamma+2\varepsilon y+\zeta yy} \quad \text{et} \quad y = \frac{-\beta-\delta x-\varepsilon xx+X}{\gamma+2\varepsilon x+\zeta xx}$$

ideoque

$$Y = \beta+\delta y+\varepsilon yy+x(\gamma+2\varepsilon y+\zeta yy),$$

$$X = \beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx).$$

7. Hanc aequationem canonicam differentiemus ac prodibit aequatio differentialis per binarium divisa

$$0 = +\beta dx + \gamma x dx + \delta y dx + 2\varepsilon xy dx + \varepsilon yy dx + \zeta xyy dx$$
$$+\beta dy + \gamma y dy + \delta x dy + 2\varepsilon xy dy + \varepsilon xx dy + \zeta xxy dy;$$

quae cum reducatur ad hanc formam

$$0 = +dx(\beta + \delta y + \varepsilon yy) + x dx(\gamma + 2\varepsilon y + \zeta yy)$$
$$+dy(\beta + \delta x + \varepsilon xx) + y dy(\gamma + 2\varepsilon x + \zeta xx),$$

quoniam coefficientes ipsorum dx et dy sunt eae ipsae quantitates, quas modo pro formulis radicalibus X et Y exhibuimus, ista aequatio differentialis erit

$$0 = Ydx + Xdy \quad \text{seu} \quad \frac{dx}{X} + \frac{dy}{Y} = 0;$$

in qua cum variabiles x et y sint separatae, si quidem pro X et Y valores illos surdos substituamus, per integrationem inde hanc aequationem finitam obtinebimus

$$\int\frac{dx}{X} + \int\frac{dy}{Y} = \text{Const.}$$

8. Cum igitur haec aequatio integralis certam quandam relationem inter variabiles x et y exprimat, ea a relatione in aequatione contenta diversa esse non potest sicque ipsa aequatio canonica continebit istam aequationem integralem. Etsi ergo in aequatione differentiali $\frac{dx}{X} + \frac{dy}{Y} = 0$ neutra pars est integrabilis atque adeo neque per circuli quadraturam neque logarithmos expediri potest, tamen integratio algebraicam relationem inter ambas variabiles x et y praebet, propterea quod haec aequatio integrata cum ipsa aequatione canonica convenit. Quin etiam dico aequationem canonicam non solum casum particularem integralis praebere, cuiusmodi casus saepe aequationibus maxime complicatis satisfaciunt, sed eam adeo integrale completum secundum omnem extensionem exhibere.

9. Ad hoc ostendendum, in quo sine dubio summa vis huius integrationis agnosci debet, notasse sufficit in aequatione canonica una constante plus contineri quam in aequatione differentiali. Vidimus enim aequationem canoni-

cam quinque involvere constantes arbitrarias; unde examinemus, quot huiusmodi constantes aequatio differentialis complectatur. Manifestum autem est eam huiusmodi habere formam

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}+\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=0,$$

in qua quidem etiam quinque constantes A, B, C, D, E inesse videntur; verum evidens est unamquamque per divisionem tolli posse, ita ut re vera quatuor tantum inesse sint censendae. Quare cum aequatio integralis quinque contineat, una arbitrio nostro relinquitur, quod est manifestum indicium integralis completi.

10. Utcunque autem isti quinque coefficientes A, B, C, D, E se habeant, semper coefficientes aequationis canonicae iis conformiter ita definiri possunt, ut unus maneat indeterminatus. Dividamus enim aequationem differentialem per quantitatem indefinitam p, quae iam sublata est censenda, ut re vera fuerit

$$X=\sqrt{(Ap+2Bpx+Cpxx+2Dpx^3+Epx^4)}.$$

Iam evolvamus quoque secundum potestates ipsius x valorem primitivum ipsius X, qui erit

$$X=\sqrt{\left(\begin{matrix}\beta\beta\\-\alpha\gamma\end{matrix}\;\left.\begin{matrix}+2\beta\delta\\-2\alpha\varepsilon\\-2\beta\gamma\end{matrix}\right\}x\;\left.\begin{matrix}+2\beta\varepsilon\\+\ \delta\delta\\-\ \alpha\zeta\\-4\beta\varepsilon\\-\ \gamma\gamma\end{matrix}\right\}x^2\;\left.\begin{matrix}+2\delta\varepsilon\\-2\beta\zeta\\-2\gamma\varepsilon\end{matrix}\right\}x^3\;\left.\begin{matrix}+\varepsilon\varepsilon\\-\gamma\zeta\end{matrix}\right\}x^4\right)},$$

atque istae litterae α, β, γ, δ, ε, ζ ita definiantur, ut haec forma cum priori congruens reddatur; sic enim patebit unam determinationem adhuc arbitrio nostro relinqui.

11. Satisfieri igitur oportet sequentibus quinque aequationibus

$$\text{I. } \beta\beta-\alpha\gamma=Ap$$

$$\text{II. } \beta\delta-\alpha\varepsilon-\beta\gamma=Bp$$

$$\text{III. } \delta\delta-\alpha\zeta-2\beta\varepsilon-\gamma\gamma=Cp$$

$$\text{IV. } \delta\varepsilon-\beta\zeta-\gamma\varepsilon=Dp$$

$$\text{V. } \varepsilon\varepsilon-\gamma\zeta=Ep.$$

Ponamus ad abbreviandum $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$ et incipiamus a II et IV

$$\text{II.}\quad \beta\lambda - \alpha\varepsilon = Bp \quad \text{et} \quad \text{IV.}\quad \varepsilon\lambda - \beta\zeta = Dp,$$

unde definiemus β et ε, ita ut sit

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta}p \quad \text{et} \quad \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta}p.$$

At I et V coniunctae dant

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = Ap\zeta - Ep\alpha = \frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta}pp,$$

unde eruitur

$$p = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha},$$

qui valor in alterutra substitutus praebet

$$\gamma = \frac{(A\zeta - E\alpha)((ADD - BBE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha\alpha)}{(BB\zeta - DD\alpha)^2}.$$

12. Superest igitur III aequatio, quae ob $\delta = \gamma + \lambda$ transit in

$$2\gamma\lambda + \lambda\lambda - \alpha\zeta - 2\beta\varepsilon = Cp.$$

Cum nunc substituto valore ipsius p sit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \quad \text{et} \quad \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

si isti valores pro γ, β, ε et p substituantur, tota aequatio per $\lambda\lambda - \alpha\zeta$ dividi poterit, quo facto reperietur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - E\alpha)(ADD - BBE)}.$$

Quoniam igitur nunc omnibus conditionibus est satisfactum, arbitrio nostro adhuc relinquuntur duo coefficientes α et ζ seu potius eorum ratio mutua, quam ergo pro lubitu definire licet. Ex quo manifestum est in aequatione integrali seu ipsa canonica inesse constantem arbitrariam ab aequatione differentiali non pendentem.

ALIA RESOLUTIO EARUNDEM FORMULARUM

13. Quia istorum valorum applicatio fieri nequit casibus, quibus

$$ADD - BBE = 0,$$

aliam resolutionem huic incommodo non obnoxiam tradam. Posito autem $\delta = \gamma + \lambda$ statuo porro

$$\lambda\lambda - \alpha\zeta = \mu \quad \text{seu} \quad \lambda\lambda = \mu + \alpha\zeta$$

atque ut ante ex aequationibus II et IV habebimus

$$\beta = \frac{p}{\mu}(D\alpha + B\lambda), \quad \varepsilon = \frac{p}{\mu}(B\zeta + D\lambda).$$

Tum vero, quia I et V coniunctae dant

$$A\zeta - E\alpha = (BB\zeta - DD\alpha)\frac{p}{\mu},$$

hinc definio rationem inter α et ζ, seu quoniam alterutram pro lubitu accipere licet, utramque hoc modo, ut sit

$$\alpha = \mu A - BBp \quad \text{et} \quad \zeta = \mu E - DDp$$

hincque

$$\lambda\lambda = \mu + (\mu A - BBp)(\mu E - DDp).$$

At alterutra I et V valoribus hactenus inventis substitutis praebebit

$$\gamma = \frac{pp}{\mu\mu}(2BD\lambda + (ADD + BBE)\mu) - \frac{2BBDDp^3}{\mu\mu} - \frac{p}{\mu}.$$

14. Quodsi iam hi valores in aequatione III substituantur, ea ad formam quidem admodum prolixam reducitur; verum negotium commodius absolvetur, si valores pro α et ζ inventi in formula ultima praecedentis resolutionis substituantur; tum enim prodibit

$$\lambda = \frac{\mu\mu}{2p} + BDp - \frac{1}{2}C\mu,$$

cuius quadratum cum superiori ipsius $\lambda\lambda$ valore coaequatum praebet

$$\mu(\mu - Cp)^2 + 4(BD - AE)pp\mu + 4(ADD - BCD + BBE)p^3 = 4pp;$$

ad quam resolvendam ponamus $\mu = pM$ eritque

$$p = \frac{4}{M(M-C)^2 + 4M(BD-AE) + 4(ADD-BCD+BBE)}$$

et

$$\mu = \frac{4M}{M(M-C)^2 + 4M(BD-AE) + 4(ADD-BCD+BBE)}$$

atque iam M est constans illa arbitraria integrale reddens completum.

15. Hoc modo omnes coefficientes α, β, γ, δ etc. eodem denominatore affecti prodibunt, qui ergo, si per eundem multiplicentur, sequenti modo sese habebunt

$$\alpha = 4(AM - BB), \quad \beta = 2B(M-C) + 4AD, \quad \gamma = 4AE - (M-C)^2,$$
$$\zeta = 4(EM - DD), \quad \varepsilon = 2D(M-C) + 4BE, \quad \delta = MM - CC + 4(AE + BD),$$

ac si illum denominatorem brevitatis gratia statuamus

$$M(M-C)^2 + 4M(BD-AE) + 4(ADD - BCD + BBE) = \varDelta,$$

aequatio nostra canonica

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

resoluta dabit

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = \mp 2\sqrt{\varDelta}(A + 2Bx + Cxx + 2Dx^3 + Ex^4),$$
$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = \mp 2\sqrt{\varDelta}(A + 2By + Cyy + 2Dy^3 + Ey^4)$$

simulque est integrale completum huius aequationis differentialis

$$0 = \frac{dx}{\pm\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{dy}{\pm\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}},$$

quia constantem arbitrariam M involvit, quae in aequationem differentialem non ingreditur.

INVESTIGATIO CASUUM QUIBUS FORMULA $\frac{Pdx}{X} + \frac{Qdy}{Y}$ FIT INTEGRABILIS

16. Designat hic P functionem ipsius x et Q similem functionem ipsius y, et quia haec formula integrabilis esse debet, sit V eius integrale, ut

habeamus

$$\frac{Pdx}{X}+\frac{Qdy}{Y}=dV \quad \text{et} \quad \int\frac{Pdx}{X}+\int\frac{Qdy}{Y}=V.$$

Cum autem sit

$$\frac{dx}{X}+\frac{dy}{Y}=0 \quad \text{ideoque} \quad \frac{dy}{Y}=\frac{-dx}{X},$$

erit

$$dV=\frac{(P-Q)dx}{X}=\frac{(P-Q)dx}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}.$$

Casus ergo investigari oportet, quibus haec formula integrationem admittit.

17. Quoniam vero nulla est ratio, cur hic differentiale dx potius insit quam dy, tertiam variabilem introducamus, quae ad utramque aequaliter referatur, siquidem quantitas V utramque aequaliter involvere debet. Statuamus ergo $x+y=s$ et in aequatione differentiali (§ 7) pro dy scribamus $ds-dx$ sicque prodibit

$$\begin{aligned}0=&+dx(\beta+\delta y+\varepsilon yy)+xdx(\gamma+2\varepsilon y+\zeta yy)\\&-dx(\beta+\delta x+\varepsilon xx)-ydx(\gamma+2\varepsilon x+\zeta xx)\\&+ds(\beta+\delta x+\varepsilon xx)+yds(\gamma+2\varepsilon x+\zeta xx),\end{aligned}$$

unde dx per ds ita definietur, ut sit

$$dx=\frac{ds(\beta+\delta x+\varepsilon xx)+yds(\gamma+2\varepsilon x+\zeta xx)}{\delta(x-y)+\varepsilon(xx-yy)-\gamma(x-y)+\zeta xy(x-y)}$$

sive

$$dx=\frac{ds}{x-y}\cdot\frac{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}{\delta-\gamma+\varepsilon(x+y)+\zeta xy},$$

quo valore substituto fiet

$$dV=\frac{(P-Q)ds}{(x-y)(\delta-\gamma+\varepsilon(x+y)+\zeta xy)}.$$

18. Cum P et Q sint similes functiones ipsarum x et y, manifestum est $P-Q$ per $x-y$ fore divisibile et fractionem $\frac{P-Q}{x-y}$ utramque variabilem x et y aequaliter esse complexuram. Quia vero posuimus $x+y=s$, ponamus insuper $xy=t$, ut sit

$$dV=\frac{P-Q}{x-y}\cdot\frac{ds}{\delta-\gamma+\varepsilon s+\zeta t}.$$

At ob $xx + yy = ss - 2t$ aequatio canonica induet hanc formam

$$0 = \alpha + 2\beta s + \gamma ss + 2(\delta - \gamma)t + 2\varepsilon st + \zeta tt,$$

ex qua elicitur

$$t = \frac{-\delta + \gamma - \varepsilon s + \sqrt{((\delta-\gamma)^2 - \alpha\zeta + 2(\delta-\gamma)\varepsilon s - 2\beta\zeta s + \varepsilon\varepsilon ss - \gamma\zeta ss)}}{\zeta},$$

ita ut sit

$$\delta - \gamma + \varepsilon s + \zeta t = \sqrt{\left((\delta-\gamma)^2 - \alpha\zeta + 2((\delta-\gamma)\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss\right)}.$$

Statuamus hanc formulam irrationalem

$$\sqrt{\left((\delta-\gamma)^2 - \alpha\zeta + 2((\delta-\gamma)\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss\right)} = S,$$

ut sit

$$t = \frac{-(\delta-\gamma) - \varepsilon\varepsilon + S}{\zeta} \quad \text{et} \quad dV = \frac{P-Q}{x-y}\cdot\frac{ds}{S}.$$

19. Ut hinc iam casus integrabilitatis eruamus, ponamus

$$P = a + bx + cxx + dx^3 + ex^4,$$
$$Q = a + by + cyy + dy^3 + ey^4$$

eritque

$$\frac{P-Q}{x-y} = b + c(x+y) + d(xx + xy + yy) + e(x^3 + xxy + xyy + y^3)$$

sive introductis novis variabilibus s et t

$$\frac{P-Q}{x-y} = b + cs + d(ss - t) + es(ss - 2t).$$

At pro t valore substituto habebimus ob $\lambda = \delta - \gamma$

$$\frac{P-Q}{x-y} = b + cs + dss + es^3 + \frac{\lambda d}{\zeta} + \frac{\varepsilon ds}{\zeta} + \frac{2\varepsilon ess}{\zeta} + \frac{(d+2es)S}{\zeta} + \frac{2\lambda es}{\zeta},$$

unde consequimur

$$dV = \frac{\zeta b + \lambda d + (\zeta c + \varepsilon d + 2\lambda e)s + (\zeta d + 2\varepsilon e)ss + \zeta es^3}{\zeta S}ds - \frac{(d+2es)ds}{\zeta},$$

quam formulam integrabilem esse oportet.

20. Quo hoc facilius praestemus, recordemur ex § 13 et 14 esse

$$(\delta-\gamma)^2-\alpha\zeta=\lambda\lambda-\alpha\zeta=\mu,\quad (\delta-\gamma)\varepsilon-\beta\zeta=Dp \quad \text{et} \quad \varepsilon\varepsilon-\gamma\zeta=Ep,$$

unde fit

$$S=\sqrt{(\mu+2Dps+Epss)}$$

sive ex § 14 et 15

$$S=\frac{2\sqrt{(M+2Ds+Ess)}}{\sqrt{\varDelta}}.$$

Ponamus porro brevitatis gratia

$$b+\frac{\lambda d}{\xi}=h,\quad c+\frac{\varepsilon d+2\lambda e}{\xi}=g,\quad d+\frac{2\varepsilon e}{\xi}=f,$$

ut sit

$$dV=\frac{(h+gs+fss+es^3)ds\sqrt{\varDelta}}{2\sqrt{(M+2Ds+Ess)}}-\frac{(d+2es)ds}{\xi};$$

statuatur partis prioris integrale

$$(\mathfrak{F}+\mathfrak{G}s+\mathfrak{H}ss)\sqrt{\varDelta(M+2Ds+Ess)}$$

eritque differentialium comparatione instituta

$$h=2\mathfrak{G}M+2\mathfrak{F}D,\quad g=4\mathfrak{H}M+6\mathfrak{G}D+2\mathfrak{F}E,$$
$$f=10\mathfrak{H}D+4\mathfrak{G}E,\quad e=6\mathfrak{H}E,$$

unde pro integrabilitate requiritur, ut sit

$$0=eD(3EM-5DD)+fE(3DD-EM)-2gDEF+2hE^3.$$

21. Hac autem conditione impleta erit

$$\mathfrak{F}=\frac{h}{2D}-\frac{fM}{4DE}+\frac{5eM}{12EE},\quad \mathfrak{G}=\frac{f}{4E}-\frac{5eD}{12EE},\quad \mathfrak{H}=\frac{e}{6E}$$

et integrale quaesitum reperietur

$$V=(\mathfrak{F}+\mathfrak{G}s+\mathfrak{H}ss)\sqrt{\varDelta(M+2Ds+Ess)}-\frac{(d+es)s}{\xi}$$

vel

$$V=\frac{1}{2}(\mathfrak{F}+\mathfrak{G}s+\mathfrak{H}ss)\varDelta S-\frac{(d+es)s}{\xi}.$$

Cum nunc sit $S = \lambda + \varepsilon(x+y) + \zeta xy$, si pro s scribamus $x+y$, valor integralis V ita per x et y exprimetur, ut sit

$$V = \frac{1}{2}\varDelta(\mathfrak{F} + \mathfrak{G}(x+y) + \mathfrak{H}(x+y)^2)(\lambda + \varepsilon(x+y) + \zeta xy) - \frac{d(x+y) + e(x+y)^2}{\xi}.$$

Quare ut pro V prodeat quantitas algebraica, coefficientes b, c, d et e non pro lubitu assumere licet, sed certam quandam relationem inter eos statui oportet, quae ultima aequalitate paragraphi praecedentis exprimitur. Ceterum hic assumsi non esse $E = 0$; si enim esset $E = 0$, valor ipsius V semper algebraice exhiberi posset, uti ex elementis integrationis est manifestum.

22. Verum si coefficientes b, c, d, e etc. utcunque assumamus, tum expressio

$$\int \frac{P\,dx}{X} + \int \frac{Q\,dy}{Y}$$

non quidem semper algebraice exhiberi poterit, attamen eius valor altiorem quadraturam non involvet quam in formula

$$\int \frac{ds}{\sqrt{(M + 2Ds + Ess)}}$$

contentam, quae propterea semper vel per logarithmos vel per arcus circulares exhiberi poterit. Cum igitur sit

$$X = \sqrt{p}(A + 2Bx + Cxx + 2Dx^3 + Ex^4) \quad \text{et} \quad \sqrt{p} = \frac{2}{\sqrt{\varDelta}},$$

erit

$$X = \frac{2}{\sqrt{\varDelta}}\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)},$$

unde invento valore ipsius V habebitur sequens integratio

$$\int \frac{dx(a + bx + cxx + dx^3 + ex^4)}{\sqrt{(A + 2Bx + Cx^2 + 2Dx^3 + Ex^4)}} + \int \frac{dy(a + by + cyy + dy^3 + ey^4)}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = \frac{2V}{\sqrt{\varDelta}}.$$

At substitutis superioribus valoribus erit

$$\frac{2V}{\sqrt{\varDelta}} = \int \frac{\xi b + \lambda d + (\xi e + \varepsilon d + 2\lambda e)s + (\xi d + 2\varepsilon e)ss + \xi e s^3}{\xi\sqrt{(M + 2Ds + Ess)}}\,ds - \frac{(d + es)s\sqrt{\varDelta}}{2(EM - DD)}$$

existente $s = x + y$. Atque hinc sequentia problemata resolvi poterunt.

PROBLEMA 1

23. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

SOLUTIO

Statim apparet huic aequationi differentiali satisfacere casum $y = x$, qui autem non nisi integrale particulare largitur. Verum ad integrale completum inveniendum, quod praeter constantes A, B, C, D, E novam constantem arbitrariam M involvat, ponamus secundum § 15 brevitatis gratia

$$\alpha = 4(AM - BB),\quad \beta = 2B(M - C) + 4AD,\quad \gamma = 4AE - (M - C)^2,$$
$$\zeta = 4(EM - DD),\quad \varepsilon = 2D(M - C) + 4BE,\quad \delta = MM - CC + 4(AE + BD)$$

atque aequatio integralis completa erit

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy,$$

quae ergo est algebraica. Hinc autem sive y per x sive vicissim x per y sequenti modo definietur posito item brevitatis ergo

$$\Delta = M(M - C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD,$$

ut sit vel

$$y = \frac{-\beta - \delta x - \varepsilon xx \pm 2\sqrt{\Delta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}}{\gamma + 2\varepsilon x + \zeta xx}$$

vel

$$x = \frac{-\beta - \delta y - \varepsilon yy \pm 2\sqrt{\Delta(A + 2By + Cyy + 2Dy^3 + Ey^4)}}{\gamma + 2\varepsilon y + \zeta yy},$$

scilicet ratione signorum ambiguorum in utraque expressione vel signa superiora vel inferiora capi debent, ita ut, si in altera formulae surdae tribuatur signum $+$, in altera formulae surdae signum $-$ tribui debeat. Quae ratio ex § 15 intelligitur, ubi in aequatione differentiali formulis surdis signa ambigua sunt adiuncta.

COROLLARIUM 1

24. Quanquam igitur aequationis differentialis propositae, in qua ambae variabiles x et y a se invicem sunt separatae, neutrum membrum integrationem absolutam admittit atque adeo neque per logarithmos neque arcus circulares in genere exprimi potest, tamen vera relatio inter variabiles x et y aequatione algebraica exhiberi potest.

COROLLARIUM 2

25. Quemadmodum scilicet, si duo arcus quantitate constante differunt, etsi neuter algebraice exprimitur, tamen eorum sinus inter se algebraicam tenent rationem, quae satisfacit aequationi differentiali

$$\frac{dy}{\sqrt{(1-yy)}}=\frac{dx}{\sqrt{(1-xx)}},$$

ita quoque aequationis differentialis propositae multoque latius patentis integrale completum algebraice exhiberi potest.

SCHOLION

26. Vis huius solutionis facilius percipietur, si eam ad casus magis restrictos applicemus, inter quos ii praecipue sunt notatu digni, ubi signum radicale vel unico vel duobus tantum terminis praefigitur, ac si unicus tantum terminus reperiatur, ratio per se est manifesta.

I. Sit enim $B=0$, $C=0$, $D=0$ et $E=0$, ut integranda sit aequatio

$$\frac{dy}{\sqrt{A}}=\frac{dx}{\sqrt{A}} \quad \text{sive} \quad dy=dx;$$

erit

$$\alpha=4AM, \quad \beta=0, \quad \gamma=-MM, \quad \delta=MM, \quad \varepsilon=0, \quad \zeta=0$$

ideoque aequatio integralis

$$0=4AM-MM(xx+yy)+2MMxy$$

seu

$$x-y=2\sqrt{\frac{A}{M}} \quad \text{vel} \quad y=x\pm\text{Const.}$$

II. Sit $A = 0$, $C = 0$, $D = 0$ et $E = 0$, ut integranda sit aequatio

$$\frac{dy}{\sqrt{2By}} = \frac{dx}{\sqrt{2Bx}} \quad \text{seu} \quad \frac{dy}{\sqrt{y}} = \frac{dx}{\sqrt{x}};$$

erit

$$\alpha = -4BB, \quad \beta = 2BM, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = 0 \quad \text{et} \quad \zeta = 0$$

ideoque aequatio integralis ob $\Delta = M^3$

$$0 = -4BB + 4BM(x+y) - MM(xx+yy) + 2MMxy$$

seu

$$y = \frac{-2BM - MMx \pm 2\sqrt{2BM^3x}}{-MM} = x + \frac{2B}{M} \pm 2\sqrt{\frac{2B}{M}x}$$

hincque $\sqrt{y} = \sqrt{x} + \text{Const.}$, uti est perspicuum.

III. Sit $A = 0$, $B = 0$, $D = 0$ et $E = 0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{Cyy}} = \frac{dx}{\sqrt{Cxx}} \quad \text{seu} \quad \frac{dy}{y} = \frac{dx}{x};$$

erit

$$\alpha = 0, \quad \beta = 0, \quad \gamma = -(M-C)^2, \quad \delta = MM - CC, \quad \varepsilon = 0 \quad \text{et} \quad \zeta = 0$$

ideoque aequatio integralis

$$0 = -(M-C)^2(xx+yy) + 2(MM-CC)xy \quad \text{seu} \quad y = nx.$$

IV. Sit $A = 0$, $B = 0$, $C = 0$ et $E = 0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{2Dy^3}} = \frac{dx}{\sqrt{2Dx^3}} \quad \text{seu} \quad \frac{dy}{y\sqrt{y}} = \frac{dx}{x\sqrt{x}};$$

erit

$$\alpha = 0, \quad \beta = 0, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = 2DM, \quad \zeta = -4DD$$

ideoque aequatio integralis

$$0 = -MM(xx+yy) + 2MMxy + 4DMxy(x+y) - 4DDxxyy,$$

quae ob $\Delta = M^3$ dat

$$y = \frac{-MMx - 2DMxx \pm 2\sqrt{2DM^3x^3}}{-MM + 4DMx - 4DDxx}$$

seu

$$\sqrt{y} = \frac{M \pm \sqrt{2DMx}}{M - 2Dx}\sqrt{x} = \frac{\sqrt{Mx}}{\sqrt{M} \pm \sqrt{2Dx}}$$

vel

$$\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{x}} \pm \sqrt{\frac{2D}{M}},$$

uti rei natura postulat.

V. Sit $A = 0$, $B = 0$, $C = 0$ et $D = 0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{Ey^4}} = \frac{dx}{\sqrt{Ex^4}} \quad \text{seu} \quad \frac{dy}{yy} = \frac{dx}{xx};$$

erit

$$\alpha = 0, \quad \beta = 0, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = 0 \quad \text{et} \quad \zeta = 4EM$$

ideoque aequatio integralis

$$0 = -MM(xx + yy) + 2MMxy + 4EMxxyy$$

hincque

$$y - x = 2xy\sqrt{\frac{E}{M}} \quad \text{seu} \quad \frac{1}{y} = \frac{1}{x} \pm 2\sqrt{\frac{E}{M}}.$$

Quando autem signum radicale complectitur duos terminos, varios casus, qui huc pertinent, sequentibus exemplis evolvemus.

EXEMPLUM 1

27. Si sit $C = 0$, $D = 0$ et $E = 0$, ut integranda sit aequatio

$$\frac{dy}{\sqrt{(A + 2By)}} = \frac{dx}{\sqrt{(A + 2Bx)}},$$

invenire aequationem integralem completam.

Erit ergo

$$\alpha = 4(AM - BB), \quad \beta = 2BM, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = 0, \quad \zeta = 0,$$

unde aequatio integralis

$$0 = 4(AM - BB) + 4BM(x + y) - MM(xx + yy) + 2MMxy$$

et ob $\Delta = M^3$

$$y = \frac{-2BM - MMx \pm 2\sqrt{M^3(A+2Bx)}}{-MM} = \frac{2B + Mx}{M} \mp 2\sqrt{\frac{A+2Bx}{M}}.$$

Unde ponendo $A = f$, $2B = g$ et $M = c$ sequitur

THEOREMA 1

28. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gy)}} = \frac{dx}{\sqrt{(f+gx)}}$$

integrale completum est

$$0 = 4cf - gg + 2cg(x+y) - cc(xx+yy) + 2ccxy,$$

unde fit

$$y = x + \frac{g}{c} \mp 2\sqrt{\frac{f+gx}{c}} \quad \text{et} \quad x = y + \frac{g}{c} \pm 2\sqrt{\frac{f+gy}{c}}.$$

EXEMPLUM 2

29. Si sit $B = 0$, $D = 0$ et $E = 0$, ut integranda sit aequatio

$$\frac{dy}{\sqrt{(A+Cyy)}} = \frac{dx}{\sqrt{(A+Cxx)}},$$

invenire aequationem integralem completam.

Erit ergo

$$\alpha = 4AM, \quad \beta = 0, \quad \gamma = -(M-C)^2, \quad \delta = MM - CC, \quad \varepsilon = 0 \quad \text{et} \quad \zeta = 0,$$

unde aequatio integralis quaesita erit

$$0 = 4AM - (M-C)^2(xx+yy) + 2(MM-CC)xy,$$

et ob $\Delta = M(M-C)^2$ erit

$$y = \frac{-(MM-CC)x \pm 2(M-C)\sqrt{M(A+Cxx)}}{-(M-C)^2} = \frac{(M+C)x \mp 2\sqrt{M(A+Cxx)}}{M-C}.$$

Quare ponendo $A = f$, $C = g$ et $M = c$ sequitur

THEOREMA 2

30. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gyy)}}=\frac{dx}{\sqrt{(f+gxx)}}$$

integrale completum est

$$0=4cf-(c-g)^2(xx+yy)+2(cc-gg)xy,$$

unde fit

$$y=\frac{(c+g)x\pm 2\sqrt{c(f+gxx)}}{c-g} \quad \text{et} \quad x=\frac{(c+g)y\mp 2\sqrt{c(f+gyy)}}{c-g}.$$

EXEMPLUM 3

31. Si sit $B=0$, $C=0$ et $E=0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{(A+2Dy^3)}}=\frac{dx}{\sqrt{(A+2Dx^3)}},$$

invenire aequationem integralem completam.

Erit ergo

$$\alpha=4AM,\quad \beta=4AD,\quad \gamma=-M^2,\quad \delta=M^2,\quad \varepsilon=2DM \quad \text{et} \quad \zeta=-4DD,$$

unde aequatio integralis quaesita est

$$0=4AM+8AD(x+y)-M^2(xx+yy)+2M^2xy+4DMxy(x+y)-4DDxxyy,$$

et cum sit $\Delta=M^3+4ADD$, erit

$$y=\frac{-4AD-MMx-2DMxx\pm 2\sqrt{(M^3+4ADD)(A+2Dx^3)}}{-MM+4DMx-4DDxx}$$

sive

$$y=\frac{4AD+MMx+2DMxx\pm 2\sqrt{(M^3+4ADD)(A+2Dx^3)}}{(M-2Dx)^2}.$$

Quare si ponatur $A=f$, $2D=g$ et $M=c$, sequitur

THEOREMA 3

32. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gy^3)}}=\frac{dx}{\sqrt{(f+gx^3)}}$$

integrale completum est

$$0=4cf+4fg(x+y)-cc(xx+yy)+2ccxy+2cgxy(x+y)-ggxxyy,$$

unde fit

$$y=\frac{2fg+ccx+cgxx\pm2\sqrt{(c^3+fgg)(f+gx^3)}}{(c-gx)^2}$$

et

$$x=\frac{2fg+ccy+cgyy\mp2\sqrt{(c^3+fgg)(f+gy^3)}}{(c-gy)^2}.$$

EXEMPLUM 4

33. Si sit $B=0$, $C=0$ et $D=0$, ut aequatio integranda sit

$$\frac{dy}{\sqrt{(A+Ey^4)}}=\frac{dx}{\sqrt{(A+Ex^4)}},$$

invenire aequationem integralem completam.

Erit ergo

$\alpha=4AM,\ \beta=0,\ \gamma=4AE-MM,\ \delta=MM+4AE,\ \varepsilon=0$ et $\zeta=4EM,$

unde aequatio integralis quaesita est

$$0=4AM+(4AE-MM)(xx+yy)+2(4AE+MM)xy+4EMxxyy,$$

et cum sit $\Delta=M^3-4AEM$, erit

$$y=\frac{-(MM+4AE)x\pm2\sqrt{M(MM-4AE)(A+Ex^4)}}{4AE-MM+4EMxx}.$$

Quare si ponatur $A=f$, $E=g$ et $M=2c$, sequitur

THEOREMA 4

34. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gy^4)}}=\frac{dx}{\sqrt{(f+gx^4)}}$$

integrale completum est

$$0=2cf-(cc-fg)(xx+yy)+2(cc+fg)xy+2cgxxyy,$$

unde fit

$$y=\frac{+(cc+fg)x\pm\sqrt{2c(cc-fg)(f+gx^4)}}{cc-fg-2cgxx}$$

et

$$x=\frac{+(cc+fg)y\mp\sqrt{2c(cc-fg)(f+gx^4)}}{cc-fg-2cgyy}.$$

EXEMPLUM 5

35. Si sit $A=0$, $C=0$ et $D=0$, ut integranda sit haec aequatio

$$\frac{dy}{\sqrt{(2By+Ey^4)}}=\frac{dx}{\sqrt{(2Bx+Ex^4)}},$$

invenire aequationem integralem completam.

Erit ergo

$$\alpha=-4BB,\quad \beta=2BM,\quad \gamma=-MM,\quad \delta=MM,\quad \varepsilon=4BE\quad \text{et}\quad \zeta=4EM$$

hincque aequatio integralis quaesita

$$\begin{aligned}0=-4BB+4BM(x+y)-MM(xx+yy)+2MMxy\\+8BExy(x+y)+4EMxxyy,\end{aligned}$$

et cum sit $\varDelta=M^3+4BBE$, erit

$$y=\frac{2BM+MMx+4BExx\pm2\sqrt{(M^3+4BBE)(2Bx+Ex^4)}}{MM-8BEx-4EMxx}.$$

Quare si ponatur $2B=f$, $E=g$, $M=c$, $x=xx$ et $y=yy$, sequitur

THEOREMA 5

36. *Huius aequationis differentialis*

$$\frac{dy}{\sqrt{(f+gy^6)}}=\frac{dx}{\sqrt{(f+gx^6)}}$$

integrale completum est

$$0=-ff+2cf(xx+yy)-cc(x^4+y^4)+2ccxxyy+4fgxxyy(xx+yy)+4cgx^4y^4,$$

unde fit

$$yy=\frac{cf+ccxx+2fgx^4\pm 2x\sqrt{(c^3+ffg)(f+gx^6)}}{cc-4fgxx-4cgx^4}$$

et

$$xx=\frac{cf+ccyy+2fgy^4\mp 2y\sqrt{(c^3+ffg)(f+gy^6)}}{cc-4fgyy-4cgy^4}.$$

SCHOLION 1

37. Probabile hinc videtur etiam huius aequationis differentialis

$$\frac{dy}{\sqrt{(f+gy^n)}}=\frac{dx}{\sqrt{(f+gx^n)}}$$

atque adeo huius latissime patentis

$$\frac{dy}{\sqrt{(a+by+cy^2+dy^3+ey^4+fy^5+\text{etc.})}}=\frac{dx}{\sqrt{(a+bx+cx^2+dx^3+ex^4+fx^5+\text{etc.})}},$$

ad quotcunque dimensiones variabiles x et y in vinculis radicalibus assurgant aequationem dari integralem completam algebraicam. Hoc enim assertum non solum verum est ostensum, quando potestates ipsarum x et y quartum ordinem non superant, sed etiam casu $n=6$, uti vidimus, priorum formularum integratio completa algebraice succedit. Interim tamen nullus adhuc modus patet pro casu $n=5$ integrale completum aequationis

$$\frac{dy}{\sqrt{(f+gy^5)}}=\frac{dx}{\sqrt{(f+gx^5)}}$$

exhibendi; multo minus id ad casus, quibus n senarium superat, extendere licet, etiamsi pro casibus $n=1$, $n=2$, $n=3$, $n=4$ et $n=6$ sit in promtu. Etsi autem de successu in reliquis casibus vix dubitare licet, tamen restrictio

necessaria videtur, ut exponens n sit numerus integer, nisi forte et eos casus fractionum adiicere lubuerit, quibus utraque formula per se fit integrabilis, uti evenit, si n sit fractio unitatem pro numeratore habens. Praeterea vero certum est veritatem nonnisi pro signo radicali quadrato subsistere posse; neque enim haec aequatio

$$\frac{dy}{\sqrt[3]{(f+gy^3)}}=\frac{dx}{\sqrt[3]{(f+gx^3)}}$$

neque haec

$$\frac{dy}{\sqrt[4]{(f+gy^4)}}=\frac{dx}{\sqrt[4]{(f+gx^4)}}$$

aliaeque harum similes integralia completa algebraica admittunt, quia hae formulae ad rationalitatem perductae tam logarithmos quam quadraturam circuli mixtim involvunt atque ex talium quantitatum heterogenearum comparatione aequatio algebraica resultare nequit. Haec eadem vero ratio dubitationem superiorem quoque decidit; ac iam audacter pronunciare possumus hanc aequationem differentialem

$$\frac{dy}{\sqrt{(a+by+cy^2+dy^3+ey^4+fy^5+gy^6)}}=\frac{dx}{\sqrt{(a+bx+cx^2+dx^3+ex^4+fx^5+gx^6)}}$$

generaliter per aequationem algebraicam integrari non posse; inde enim sequeretur integratio algebraica huius aequationis

$$\frac{dy}{A+By+Cyy+Dy^3}=\frac{dx}{A+Bx+Cxx+Dx^3},$$

quod utique esset absurdum; multo minus igitur integratio in aequationibus magis compositis succedet. Verum nequidem integrabilitas ad potestatem quintam usque extendi potest; nam posito $g=0$ si etiam statuatur $a=0$ et pro y et x scribatur yy et xx, prodit haec aequatio differentialis

$$\frac{dy}{\sqrt{(b+cy^2+dy^4+ey^6+fy^8)}}=\frac{dx}{\sqrt{(b+cx^2+dx^4+ex^6+fx^8)}},$$

in qua, si radicis extractio succedat, continebitur haec

$$\frac{dy}{A+By^2+Cy^4}=\frac{dx}{A+Bx^2+Cx^4},$$

quam in genere integrationem algebraicam non admittere est manifestum.

SCHOLION 2

38. Nunc igitur pro certo affirmare licet ex hoc genere aequationem differentialem latissime patentem, quae quidem generaliter algebraice integrari queat, esse eam ipsam, quam hactenus tractavimus

$$\frac{dy}{\sqrt{(A+2By+Cy^2+2Dy^3+Ey^4)}}=\frac{dx}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}}$$

et cuius aequationem integralem completam assignavimus. Quam ob causam haec aequatio multo magis est notatu digna, quod in hoc genere est generalissima, quae integrationem algebraicam admittat. Quoniam igitur eius integrationem iam exposui, operae pretium erit eius usum in comparatione linearum curvarum, quarum elementa per huiusmodi formulas exprimuntur, uberius ostendere, si quidem in iis omnia continentur, quae in hoc genere praestari possunt. Atque haec ipsa consideratio nos quoque ad integrationem huiusmodi aequationum

$$\frac{n\,dy}{\sqrt{(A+2By+Cy^2+2Dy^3+Ey^4)}}=\frac{m\,dx}{\sqrt{(A+2Bx+Cx^2+2Dx^3+Ex^4)}}$$

manuducet, si quidem m et n fuerint numeri integri.

PROBLEMA 2

39. *Si linea curva habeatur, cuius arcus sive abscissae sive applicatae sive cordae sive alii cuicunque rectae variabili z ad curvam relatae respondens sit*

$$=\int\frac{\mathfrak{A}\,dz}{\sqrt{(A+2Bz+Cz^2+2Dz^3+Ez^4)}},$$

deturque in hac curva arcus quicunque AB (Fig. 1), *ab alio quovis puncto P arcum abscindere PQ, qui aequalis sit illi arcui AB.*

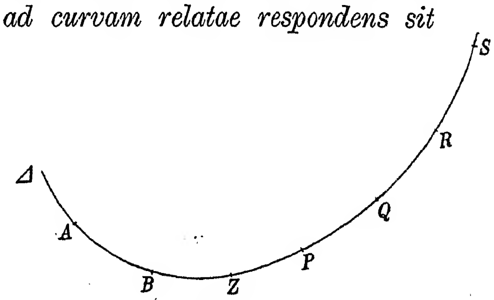

Fig. 1.

SOLUTIO

Ex coefficientibus datis A, B, C, D, E quaerantur hi alii

$$\alpha=4(AM-BA),\quad \beta=2B(M-C)+4AD,\quad \gamma=4AE-(M-C)^2,$$
$$\zeta=4(EM-DD),\quad \varepsilon=2D(M-C)+4BE,\quad \delta=MM-CC+4(AE+BD),$$

ubi M denotat novam constantem arbitrariam, atque vidimus hanc aequationem algebraicam

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy$$

congruere cum hac transcendente

$$\int \frac{\mathfrak{A}dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} - \int \frac{\mathfrak{A}dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = \text{Const.},$$

ubi quantitas constans ita definiri debet, ut illi M sit consentanea. Si iam ponamus in curva proposita variabilem z puncto Z respondere curvaeque initium in puncto $\varDelta$ statui atque ad abbreviandum hunc arcum $\varDelta Z$ ita indicemus $\varPi:z$, ut sit

$$\int \frac{\mathfrak{A}dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} = \varPi:z,$$

erit ex aequatione superiori

$$\varPi:y - \varPi:x = \text{Const.}$$

Respondeant nunc punctis A et B rectae a et b, punctis vero P et Q rectae p et q, ut sint arcus

$$\varDelta A = \varPi:a, \quad \varDelta B = \varPi:b, \quad \varDelta P = \varPi:p \quad \text{et} \quad \varDelta Q = \varPi:q$$

ideoque

$$\text{arcus } AB = \varPi:b - \varPi:a \quad \text{et} \quad \text{arcus } PQ = \varPi:q - \varPi:p,$$

ac loco x et y scribamus p et q, ut sit

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq;$$

erit $\varPi:q - \varPi:p = \text{Const.}$ Quodsi ergo constantem M ita assumamus, ut facto $p = a$ prodeat $q = b$, habebimus

$$\varPi:q - \varPi:p = \varPi:b - \varPi:a$$

ideoque arcum $PQ =$ arcui AB, uti requiritur. Constans igitur M, vel si ponamus $M - C = L$, ut sit $M = C + L$, constans L ex sequenti aequatione debet definiri

$$\begin{aligned} 0 = 4AC - 4BB + 4AL + 2(2BL + 4AD)(a+b) + (4AE - LL)(aa+bb) \\ + 2(LL + 2CL + 4AE + 4BD)ab + 2(2DL + 4BE)ab(a+b) \\ + 4(CE - DD + EL)aabb, \end{aligned}$$

unde fit

$$LL = \left\{ \begin{array}{l} \dfrac{4L(A+B(a+b)+Cab+Dab(a+b)+Eaabb)+4AC+8AD(a+b)+8(AE+BD)ab}{(b-a)^2} \\ + \dfrac{4CEaabb-4BB+4AE(aa+bb)+8BEab(a+b)-4DDaabb}{(b-a)^2} \end{array} \right\}$$

et radice extracta

$$L = \left\{ \begin{array}{l} \dfrac{2(A+B(a+b)+Cab+Dab(a+b)+Eaabb)}{(b-a)^2} \\ \pm \dfrac{2\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)}}{(b-a)^2} \end{array} \right\}$$

sicque erit

$$M = \frac{2A+2B(a+b)+C(aa+bb)+2Dab(a+b)+2Eaabb)}{(b-a)^2}$$

$$\pm \frac{2}{(b-a)^2}\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bb+Cbb+2Db^3+Eb^4\right)}.$$

Quo valore invento si iam definiantur valores coefficientium α, β, γ, δ, ε, ζ, quoniam ex dato curvae puncto P datur variabilis p, ex ea valor idoneus variabilis q, cui curvae punctum Q respondet, determinabitur per hanc aequationem

$$0 = \alpha + \beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq;$$

ex qua, si brevitatis gratia ponamus

$$\varDelta = M(M-C)^2 + 4M(BD-AE) + 4(ADD+BBE) - 4BCD,$$

habebitur

$$q = \frac{-\beta-\delta p-\varepsilon pp \pm 2\sqrt{\varDelta(A+2Bp+Cpp+2Dp^3+Ep^4)}}{\gamma+2\varepsilon p+\zeta pp}$$

sicque dato arcu AB et puncto P assignabitur punctum Q, ut arcus PQ aequalis fiat arcui AB. Reperientur autem ob signum ambiguum bina puncta Q, quorum alterum citra, alterum ultra punctum P erit situm.

COROLLARIUM 1

40. Invento valore q simili modo a puncto Q ulterius abscindi poterit arcus QR arcui AB aequalis. Posita enim variabili puncto R respondente

$= r$ capiatur

$$r = \frac{-\beta - \delta q - \varepsilon qq \pm 2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}}{\gamma + 2\varepsilon q + \zeta qq}$$

sicque a puncto P simul abscindetur arcus PR duplus arcus dati AB.

COROLLARIUM 2

41. Quoniam r hinc duplicem obtinet valorem, notandum est alterum iterum in p abire, quia ante animadvertimus esse

$$p = \frac{-\beta - \delta q - \varepsilon qq \mp 2\sqrt{\Delta(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}}{\gamma + 2\varepsilon q + \zeta qq};$$

quare, ut arcus PR evadat duplus, idem signum, quod in valore ipsius q fuerit electum, in valore ipsius r capi oportet.

COROLLARIUM 3

42. Pari modo ultra R reperietur punctum S, ut denuo arcus RS aequalis sicque angulus PS triplus evadat arcus AB; inventa enim variabili r valor variabilis s puncto S respondentis hac formula exprimetur

$$s = \frac{-\beta - \delta r - \varepsilon rr \pm 2\sqrt{\Delta(A + 2Br + Crr + 2Dr^3 + Er^4)}}{\gamma + 2\varepsilon r + \zeta rr}$$

hocque modo quousque libuerit ulterius progredi licet.

COROLLARIUM 4

43. Hac ergo repetita operatione a dato puncto P arcus abscindi poterit, qui se habeat ad arcum AB, ut numerus quicunque integer m ad unitatem. Quare si ab alio puncto abscindatur arcus, qui sit ad eundem AB ut alius numerus integer n ad unitatem, duo habebuntur arcus rationem quamcunque numeri ad numerum tenentes.

COROLLARIUM 5

44. Omnium igitur curvarum, quarum arcus variabili cuipiam z respondens huiusmodi formula

$$\int \frac{\mathfrak{A}\,dz}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

exprimitur, haec est proprietas, ut earum arcus simili modo inter se comparari possint, quo arcus circuli inter se comparare licet. Atque ob rationes supra allegatas haec similitudo cum circulo vix ad alias curvas, nisi quarum rectificatio ad hanc formulam reduci potest, extendi videtur.

EXEMPLUM

45. Proposita sit linea curva, cuius arcus ad quampiam rectam variabilem v relatus hac formula integrali $\int \frac{dv}{\sqrt{(1-v^6)}}$ exprimatur, cuiusmodi curvae algebraicae infinitae exhiberi possunt, in qua a puncto P arcus abscindi oporteat PQ, PR, PS ad datum arcum AB rationem tenentes vel aequalitatis vel duplam vel triplam.

Quia haec expressio in nostra forma generali non continetur, eo reducatur ponendo $vv = z$ seu $v = \sqrt{z}$; sic enim arcus huic novae variabili z respondens erit $= \int \frac{dz}{2\sqrt{(z-z^4)}}$. Fiat ergo $\mathfrak{A} = \frac{1}{2}$ et $A = 0$, $B = \frac{1}{2}$, $C = 0$, $D = 0$ et $E = -1$, unde obtinetur

$$\alpha = -1, \quad \beta = M, \quad \gamma = -MM, \quad \delta = MM, \quad \varepsilon = -2, \quad \zeta = -4M$$

ideoque constituta aequatione

$$0 = -1 + 2M(p+q) - MM(pp+qq) + 2MMpq - 4pq(p+q) - 4Mppqq,$$

unde fit

$$q = \frac{M + MMp - 2pp \pm 2\sqrt{(M^3-1)(p-p^4)}}{MM + 4p + 4Mpp},$$

erit

$$\int \frac{dq}{2\sqrt{(q-q^4)}} - \int \frac{dp}{2\sqrt{(p-p^4)}} = \text{Const.}$$

seu

$$\Pi : q - \Pi : p = \Pi : b - \Pi : a,$$

si quidem a, b, p, q sint valores variabilis z, qui arcubus $\varDelta A$, $\varDelta B$, $\varDelta P$ et $\varDelta Q$ conveniunt. At iam constans M ex datis a et b ita definiri debet, ut sit

$$0 = -1 + 2M(a+b) - MM(b-a)^2 - 4ab(a+b) - 4Maabb,$$

unde fit

$$M = \frac{a + b - 2aabb \pm 2\sqrt{(a - a^4)(b - b^4)}}{(b - a)^2}$$

et

$$\sqrt{(M - 1)} = \frac{\sqrt{a(1 - a)(1 + b + bb)} \pm \sqrt{b(1 - b)(1 + a + aa)}}{(b - a)},$$

$$\sqrt{(M^3 - 1)} = \frac{(a + 3b - 4ab^3)\sqrt{(a - a^4)} + (b + 3a - 4a^3b)\sqrt{(b - b^4)}}{(b - a)^3}.$$

Invento hoc modo valore constantis M ex data quantitate p invenitur q atque hinc porro valor variabilis r puncto R respondens, scilicet

$$r = \frac{M + MMq - 2qq \pm 2\sqrt{(M^3 - 1)(q - q^4)}}{MM + 4q + 4Mqq},$$

sicque a puncto P arcus quicunque multiplus arcus dati AB abscindi poterit.

SCHOLION

46. Circa huiusmodi curvas singularis affectio notari meretur; si enim brevitatis gratia ponamus

$$\sqrt{(a - a^4)} = \mathfrak{a} \quad \text{et} \quad \sqrt{(b - b^4)} = \mathfrak{b},$$

ut sit

$$M = \frac{a + b - 2aabb + 2\mathfrak{a}\mathfrak{b}}{(b - a)^2} \quad \text{et} \quad \sqrt{(M^3 - 1)} = \frac{(a + 3b - 4ab^3)\mathfrak{a} + (b + 3a - 4a^3b)\mathfrak{b}}{(b - a)^3},$$

utraque quantitas radicalis $\mathfrak{a}$ et $\mathfrak{b}$ tam affirmative quam negative capi potest, unde pro M geminus valor habetur; ex quo pro

$$q = \frac{M + MMp - 2pp \pm 2\sqrt{(M^3 - 1)(p - p^4)}}{MM + 4p + 4Mpp}$$

ob novam signi ambiguitatem quaterni valores resultant. Binos quidem natura rei ostendit, quia punctum Q tam ante quam post punctum P capi potest, sed quia quatuor reperiuntur, id indicio est curvam duplici ramo esse praeditam et in utroque arcus aequales exhiberi. Consideremus casum, quo punctum P in ipso puncto A capitur, ita ut sit $p = a$ et

$$q = \frac{M + MMa - 2aa \pm 2\mathfrak{a}\sqrt{(M^3 - 1)}}{MM + 4a + 4Maa},$$

quae forma substituto pro M valore statim duos valores praebet aequales $q=b$; at duo reliqui diversi continentur in

$$q=\frac{4a^3+9aab-6abb+b^3-4a^6-12a^4bb+8a^6b^3\pm 4a(3a-b-2a^3b)\mathfrak{ab}}{aa+6ab+bb+8a^5-24a^4b+16a^3bb-16aab^3+16a^5b^3-8a^6bb\pm 4(a+b-4a^3b+2a^4)\mathfrak{ab}},$$

qui duo valores semper sunt diversi, nisi sit vel $b=a$ vel $a=\frac{1}{1\pm\sqrt{3}}$; illo casu prodit $q=a=b$, hoc vero reperitur $q=\frac{1-b}{1+2b}$. Punctum ergo curvae, quod respondet quantitati $\frac{1}{1\pm\sqrt{3}}$, singulari proprietate erit praeditum.

PROBLEMA 3

47. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{2dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

SOLUTIO

Istud integrale quaesitum ex praecedenti problemate colligi potest. Capiatur enim punctum P in ipso puncto B, ut sit $p=b$, et consideretur tantum punctum A ut fixum, B vero seu P ut variabile, ex quo continuo assignari debeat punctum Q, ut sit arcus AQ duplus arcus AP. Posita ergo variabili p loco b sumatur

$$M=\frac{2A+2B(a+p)+C(aa+pp)+2Dap(a+p)+2Eaapp}{(p-a)^2}$$

$$\pm\frac{2}{(p-a)^2}\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)},$$

ita ut iam M sit functio variabilis p et constantis a. Deinde posito brevitatis gratia $M-C=L$ seu

$$L=\left\{\begin{array}{c}\dfrac{2(A+B(a+p)+Cap+Dap(a+p)+Eaapp)}{(p-a)^2}\\ \pm\dfrac{2\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}}{(p-a)^2}\end{array}\right\}$$

definiatur q per hanc aequationem

$$\begin{aligned}0 = 4AC - 4BB + 4AL + 2(2BL + 4AD)(p+q) + (4AE - LL)(pp+qq)\\ + 2(LL + 2CL + 4AE + 4BD)pq + 2(2DL + 4BE)pq(p+q)\\ + 4(CE - DD + EL)ppqq\end{aligned}$$

eritque ob $b = p$

$$\Pi : q - \Pi : p = \Pi : p - \Pi : a \quad \text{seu} \quad \Pi : q = 2\Pi : p - \Pi : a,$$

quae aequatio differentiata dat

$$\frac{dq}{\sqrt{(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}} = \frac{2dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}},$$

cuius propterea integralis est illa aequatio algebraica inter p et q exhibita, quam simul patet esse integralem completam, quoniam continet quantitatem constantem a, quae in aequatione differentiali non inest.

COROLLARIUM 1

48. Si retinente L valorem exhibitum inventaque variabili q per p ex q simili modo quaeratur r, ut sit

$$\Pi : r - \Pi : q = \Pi : p - \Pi : a,$$

erit

$$\Pi : r = 3\Pi : p - 2\Pi : a,$$

unde prodit aequatio differentialis

$$\frac{dr}{\sqrt{(A + 2Br + Crr + 2Dr^3 + Er^4)}} = \frac{3dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}},$$

cuius ergo aequatio integralis completa est

$$\begin{aligned}0 = 4(AC - BB + AL) + 2(2BL + 4AD)(q+r) + (4AE - LL)(qq+rr)\\ + 2(LL + 2CL + 4AE + 4BD)qr + 2(2DL + 4BE)qr(q+r)\\ + 4(CE - DD + EL)qqrr.\end{aligned}$$

COROLLARIUM 2

49. Quo haec magis contrahamus, postquam ex coefficientibus datis A, B, C, D, E et variabili p una cum constanti arbitraria a ita fuerit definita quantitas L, ut sit

$$\tfrac{1}{2}L(p-a)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$
$$\pm \sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bp + Cpp + 2Dp^3 + Ep^4)},$$

hinc determinentur sequentes coefficientes variabiles

$$\alpha = 4(AC - BB + AL), \quad \beta = 2BL + 4AD, \quad \gamma = 4AE - LL,$$
$$\zeta = 4(CE - DD + EL), \quad \varepsilon = 2DL + 4BE, \quad \delta = LL + 2CL + 4AE + 4BD.$$

COROLLARIUM 3

50. His iam quantitatibus inventis erit huius aequationis differentialis

$$\frac{dq}{\sqrt{(A + 2Bq + Cqq + 2Dq^3 + Eq^4)}} = \frac{2dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}$$

aequatio integralis completa

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq.$$

COROLLARIUM 4

51. Porro huius aequationis differentialis

$$\frac{dr}{\sqrt{(A + 2Br + Crr + 2Dr^3 + Er^4)}} = \frac{3dp}{\sqrt{(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}$$

aequatio integralis completa erit

$$0 = \alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr,$$

postquam scilicet variabilis q ope praecedentis aequationis ex p fuerit determinata.

COROLLARIUM 5

52. Simili modo progrediendo huius aequationis differentialis

$$\frac{ds}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}}=\frac{4dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}$$

aequatio integralis completa erit

$$0=\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss,$$

postquam ex praecedentibus aequationibus r per q et q per p fuerint definitae.

COROLLARIUM 6

53. Hoc modo, quousque libuerit, ulterius progredi licet sicque in genere aequatio integralis inveniri poterit completa huius differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\frac{mdp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}},$$

quicunque numerus integer pro m assumatur.

PROBLEMA 4

54. *Si m et n fuerint numeri integri quicunque, invenire aequationem integralem completam huius differentialis*

$$\frac{ndy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{mdx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

SOLUTIO

Quaeratur primum ope praeced. probl. aequatio integralis completa istius differentialis

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}=\frac{ndp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}},$$

quae erit algebraica ac praeter variabiles p et x constantem arbitrariam a

involvens. Deinde simili modo quaeratur aequatio integralis completa huius differentialis

$$\frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{m\,dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}},$$

quae etiam erit algebraica inter binas variabiles y et p insuperque constantem arbitrariam b complectetur. Ex his duabus aequationibus eliminetur variabilis p, ut obtineatur aequatio algebraica inter x et y, quae erit integralis completa huius differentialis

$$\frac{n\,dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{m\,dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

Quia autem duas constantes arbitrarias a et b continebit, alterutri pro lubitu valorem determinatum tribuere licet vel inter eas datam rationem statuere; pro integrali enim completa sufficit, ut una constans arbitraria introducatur.

SCHOLION

55. Si m et n sint numeri modice magni, nemo certe aequationem algebraicam inter x et y evolutam exhibebit; cum enim tot eliminationibus sit opus, evidens est ad aequationem plurimorum terminorum, in qua variabiles x et y ad summas dimensiones exsurgant, perveniri oportere. Atque adeo in casu problematis 3, ubi est $m=2$ et $n=1$, nemo facile eliminationis opus perficiet. Neque vero hoc etiam opus est, cum ad nostrum institutum sufficiat ostendisse aequationem integralem esse algebraicam eiusque constructionem geometrice absolvi posse; tantum enim abest, ut alienae variabiles q, r, s etc., quae in subsidium sunt vocatae, calculum turbent ideoque eliminari debeant, ut potius ad constructionem commode instituendam absolute sint necessariae.

Atque haec sunt fere, quae de curvis, quarum rectificatio hac formula

$$\int\frac{\mathfrak{A}\,dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$$

exprimitur, tradi operae pretium videbatur, quae eo redeunt, ut earum arcus inter se perinde atque arcus circulares comparari queant, siquidem proposito arcu quocunque AB a puncto dato P arcus abscindi possunt, qui ad illum rationem teneant rationalem quamcunque. Consideremus igitur etiam curvas, quarum rectificatio tali formula exprimitur

$$\int\frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

de quibus curvis quoque affectiones egregiae circa comparationem arcuum notari merentur; quem in finem evolutio formularum huc pertinentium supra § 16 et seqq. est instituta. Similis scilicet comparatio inter arcus huiusmodi curvarum suscipi potest, quae iam pridem inter arcus parabolae fieri posse est ostensa; atque inde sequentium problematum solutionem derivare licebit.

PROBLEMA 5

56. *Proposita curva, cuius arcus indefinite variabili cuipiam z respondens hac formula exprimatur*

$$\int \frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

si in ea detur arcus quicunque AB (Fig. 1, p. 341), *a dato puncto P arcum abscindere PQ, qui ab illo arcu AB differat linea sive geometrice assignabili sive a circuli hyperbolaeve quadratura pendente.*

SOLUTIO

Sit in curva proposita $\varDelta Z$ arcus variabili z respondens, qui brevitatis gratia ita exprimatur $\Pi:z$, ut sit

$$\Pi:z=\int \frac{dz(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}.$$

Punctis autem A, B, P, Q respondeant variabilis z valores a, b, p, q, ut sit

$$\varDelta A=\Pi:a,\quad \varDelta B=\Pi:b,\quad \varDelta P=\Pi:p \quad \text{et} \quad \varDelta Q=\Pi:q,$$

hincque erit

$$\text{arcus datus}\quad AB=\Pi:b-\Pi:a$$

et

$$\text{arcus quaesitus}\quad PQ=\Pi:q-\Pi:p.$$

Iam primum ex coefficientibus A, B, C, D, E et constanti arbitraria M deinceps definienda formentur quantitates sequentes

$$\alpha=4(AM-BB),\quad \beta=2B(M-C)+4AD,\quad \gamma=4AE-(M-C)^2,$$

$$\zeta=4(EM-DD),\quad \varepsilon=2D(M-C)+4BE,\quad \delta=MM-CC+4(AE+BD);$$

tum vero porro statuatur

$$\Delta = M(M-C)^2 + 4M(BD-AE) + 4(ADD+BBE) - 4BCD$$

atque inter p et q haec constituatur relatio, ut sit

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq,$$

ex qua data variabili p altera q puncto Q respondens ita definitur, ut sit

$$q = \frac{-\beta - \delta p - \varepsilon pp \pm 2\sqrt{\Delta(A + 2Bp + Cpp + 2Dp^3 + Ep^4)}}{\gamma + 2\varepsilon p + \zeta pp},$$

unde innotescet curvae punctum Q, ita ut differentia inter arcus AB et PQ sit vel geometrice assignabilis vel saltem a quadratura circuli seu hyperbolae pendeat, cuius rei ratio in indole coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ numeratoris est sita. Quomodo igitur differentia ista exprimatur, videamus; quia valorem ipsius q iam invenimus, ponamus $p+q=s$ et ex § 19 colligimus fore posito $\delta - \gamma = \lambda$

$$\Pi:q - \Pi:p = \text{Const.} - \frac{2(\mathfrak{D}+\mathfrak{E}s)s\sqrt{\Delta}}{\zeta}$$
$$+ \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}ds,$$

quod integrale manifestum est vel esse algebraicum vel a quadratura circuli hyperbolaeve pendere. Sit istud integrale brevitatis gratia $= S$; cuius valor posito $s = a+b$ fiat $= I$ et pro constante definienda statuatur $p = a$ et $q = b$ fierique debet

$$\text{Const.} = \Pi:b - \Pi:a + \frac{2(\mathfrak{D} + \mathfrak{E}(a+b))(a+b)\sqrt{\Delta}}{\zeta} - I,$$

ex quo habebitur

$$\text{arcus } PQ - \text{arcu } AB = \frac{2\mathfrak{D}(a+b) + 2\mathfrak{E}(a+b)^2}{\zeta}\sqrt{\Delta} - \frac{2\mathfrak{D}(p+q) + 2\mathfrak{E}(p+q)^2}{\zeta}\sqrt{\Delta}$$
$$- I + \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}ds.$$

At constans arbitraria M etiam ita definiri debet, ut posito $p = a$ fiat $q = b$;

quocirca erit

$$M = \frac{1}{(b-a)^2}\big(2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b) + 2Eaabb\big)$$
$$\pm \frac{2}{(b-a)^2} \sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bb + Cbb + 2Db^3 + Eb^4)}\,.$$

Hinc ergo cognita constante hac M et ex puncto P definito puncto Q differentia arcuum AB et PQ vel geometrice vel per quadraturam circuli hyperbolaeve assignari potest.

COROLLARIUM 1

57. Ex datis ergo punctis A et B seu variabilis z valoribus a et b primum constans arbitraria M ita definiatur, ut sit

$$M = \frac{1}{(b-a)^2}\big(2A + 2B(a+b) + C(aa+bb) + 2Dab(a+b) + 2Eaabb\big)$$
$$\mp \frac{2}{(b-a)^2} \sqrt{(A + 2Ba + Caa + 2Da^3 + Ea^4)(A + 2Bb + Cbb + 2Db^3 + Eb^4)}\,.$$

Tum hinc definitis modo praecepto coefficientibus α, β, γ, δ, ε, ζ ex dato puncto P punctum Q per hanc aequationem determinetur

$$0 = \alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq$$

atque arcuum PQ et AB differentia erit vel algebraica vel a circuli hyperbolaeve quadratura pendens.

COROLLARIUM 2

58. Ad istam autem arcuum differentiam assignandam capi debet posito $p + q = s$ hoc integrale, ubi $\lambda = \delta - \gamma = 2M(M - C) + 4BD$,

$$S = \int \frac{\zeta\mathfrak{B} + \lambda\mathfrak{D} + (\zeta\mathfrak{C} + \varepsilon\mathfrak{D} + 2\lambda\mathfrak{E})s + (\zeta\mathfrak{D} + 2\varepsilon\mathfrak{E})ss + \zeta\mathfrak{E}s^3}{\zeta\sqrt{(M + 2Ds + Ess)}}\,ds,$$

cuius valor posito $s = a + b$ sit $= I$, quo facto erit

$$) - \text{arc.}\, AB = \frac{2\sqrt{\varDelta}}{\zeta}\big(\mathfrak{D}(a+b) + \mathfrak{E}(a+b)^2 - \mathfrak{D}s - \mathfrak{E}ss\big) - I + S$$

$$M(M-C)^2 + 4M(BD - AE) + 4(ADD + BBE) - 4BCD.$$

COROLLARIUM 3

59. Si eveniret, ut esset $\zeta = 0$, determinatio puncti Q maneret ut ante, sed pro arcuum PQ et AB differentia assignanda recurri deberet ad primas operationes. Scilicet ex $p + q = s$ quaeratur t, ut sit

$$0 = \alpha + 2\beta s + \gamma ss + 2\lambda t + 2\varepsilon st + \zeta tt,$$

eritque

$$\text{arc. } PQ - \text{arc. } AB = 2\int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss - t) + \mathfrak{E}s(ss - 2t))\sqrt{\Delta}}{\sqrt{(\lambda\lambda - \alpha\zeta + 2(\lambda\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss)}}$$

integrali hoc ita accepto, ut evanescat posito $s = a + b$. Ubi notandum est esse

$$\sqrt{(\lambda\lambda - \alpha\zeta + 2(\lambda\varepsilon - \beta\zeta)s + (\varepsilon\varepsilon - \gamma\zeta)ss)} = 2\sqrt{\Delta(M + 2Ds + Ess)} = \lambda + \varepsilon s + \zeta t.$$

COROLLARIUM 4

60. Hinc etiam colligere licet, quaenam sit futura differentia arcuum AB et PQ, si formulae elementum curvae exhibentis numerator ad plures terminos extendatur, ut sit arcus curvae

$$\int \frac{dz(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \mathfrak{D}z^3 + \mathfrak{E}z^4 + \mathfrak{F}z^5 + \mathfrak{G}z^6 + \mathfrak{H}z^7 + \text{etc.})}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}};$$

reliquis enim manentibus ut ante erit

$$\text{arc. } PQ - \text{arc. } AB = \int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss - t) + \mathfrak{E}(s^3 - 2st) + \mathfrak{F}(s^4 - 3sst + tt) + \text{etc.})}{\sqrt{(M + 2Ds + Ess)}},$$

sequentia scilicet numeratoris membra erunt

$$\mathfrak{G}(s^5 - 4s^3t + 3stt) + \mathfrak{H}(s^6 - 5s^4t + 6sstt - t^3) + \text{etc.}$$

COROLLARIUM 5

61. Si a puncto Q simili modo abscindatur R, ut sit

$$0 = \alpha + 2\beta(q + r) + \gamma(qq + rr) + 2\delta qr + 2\varepsilon qr(q + r) + \zeta qqrr,$$

ponaturque $q + r = u$ et $qr = v$, ita ut sit

$$0 = \alpha + 2\beta u + \gamma uu + 2\lambda v + 2\varepsilon uv + \zeta vv$$

seu

$$\lambda + \varepsilon u + \zeta v = 2\sqrt{\Delta(M + 2Du + Euu)},$$

erit

$$\text{arc. } PR - 2\text{ arc. } AB = \int \frac{ds(\mathfrak{B} + \mathfrak{C}s + \mathfrak{D}(ss - t) + \mathfrak{E}(s^3 - 2st) + \text{etc.})}{\sqrt{(M + 2Ds + Ess)}}$$

$$+ \int \frac{du(\mathfrak{B} + \mathfrak{C}u + \mathfrak{D}(uu - v) + \mathfrak{E}(u^3 - 2uv) + \text{etc.})}{\sqrt{(M + 2Du + Euu)}}$$

his integralibus ita sumtis, ut evanescant posito $s = a + b$ et $u = a + b$.

COROLLARIUM 6

62. Simili modo a puncto P abscindi potest arcus PS, qui triplum dati arcus AB superet quantitate sive geometrice assignabili sive a circuli hyperbolaeve quadratura pendente, hisque casibus punctum P ita assumi poterit, ut iste excessus plane evanescat, quod quidem semper praestare licebit, si excessus sit algebraicus; sin autem sit transcendens, insuper alter terminus arcus dati A vel B huic scopo conformiter determinabitur.

NOVA SERIES INFINITA MAXIME CONVERGENS PERIMETRUM ELLIPSIS EXPRIMENS

Commentatio 448 indicis Enestroemiani
Novi commentarii academiae scientiarum Petropolitanae 18 (1773), 1774, p. 71—84
Summarium ibidem p. 13—15

SUMMARIUM

In Commentariis Academiae nostrae uti et in Actis Berolinensibus passim iam Ill. Auctor series dedit infinitas, quibus ellipsis cuiuscunque perimeter exprimitur, tam concinnas et simplices, ut dari alias adhuc commodiores vix suspicari licuerit. Haec tamen series, quam Ill. Auctor in praesenti dissertatione proponit, ceteris concinnitate sua anteferenda videtur estque plane nova. Quadrantis elliptici ponantur semiaxes a et b hisque parallelae coordinatae x et y; habebitur ex natura ellipsis

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

ex qua aequatione Ill. Auctor peringeniose totius arcus seu quartae partis perimetri longitudinem determinat. Ponatur scilicet

$$x=a\sqrt{\frac{1+z}{2}} \quad \text{et} \quad y=b\sqrt{\frac{1-z}{2}},$$

unde

$$dx=\frac{a\,dz}{2\sqrt{2(1+z)}} \quad \text{et} \quad dy=\frac{-b\,dz}{2\sqrt{2(1-z)}};$$

ex quo, si arcus ponatur $=s$, habebitur

$$ds^2=dz^2\frac{a^2+b^2-(a^2-b^2)z}{8(1-z^2)}$$

hincque

$$s=\frac{1}{2\sqrt{2}}\int dz\sqrt{\frac{a^2+b^2-(a^2-b^2)z}{1-z^2}};$$

si itaque hoc integrale ita sumatur, ut posito $x=0$ evanescat, et usque ad terminum

$x = a$ extendatur, obtinebitur quaesitus arcus ellipticus. In huius itaque formulae differentialis evolutione Ill. Auctor versatur ex eaque seriem hanc simplicem et maxime convergentem elicit

$$s = \frac{c\pi}{2\sqrt{2}}\left(1 - \frac{1\cdot 1}{4\cdot 4}n^2 - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}n^4 - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12}n^6 \text{ etc.}\right),$$

ubi

$$c = \sqrt{(a^2 + b^2)} \quad \text{et} \quad n = \frac{a^2 - b^2}{a^2 + b^2}.$$

Si sit $a = b$, quadrans hic ellipticus in circularem abit et ob $n = 0$ et $c = a\sqrt{2}$ prodit, uti quidem notissimum est, $s = \frac{a\pi}{2}$. Si vero ponatur $b = 0$, curva abit in lineam rectam alteri semiaxi aequalem; ita autem est $n = 1$ et $c = a$; unde sequens resultat aequatio

$$a = \frac{a\pi}{2\sqrt{2}}\left(1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8} \text{ etc.}\right)$$

adeoque seriei infinitae

$$1 - \frac{1\cdot 1}{4\cdot 4} - \frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8} \text{ etc.},$$

quae quidem minime convergit, adcurate assignari potest summa $\frac{2\sqrt{2}}{\pi}$. Hac occasione oblata Ill. Auctor operae pretium censet in summam huius seriei etiam a posteriori inquirere; ad quod praestandum methodo sua iam saepius explicata potissimum utitur, dum nimirum quaestionem ad aequationem differentialem revocat, cuius integrale per ipsam seriem propositam exprimatur.

1. Postquam olim multum fuissem occupatus, ut plures series infinitas, quibus cuiusque ellipsis perimeter exprimeretur, investigarem, vix eram suspicatus adhuc simpliciores atque ad calculum magis accommodatas huiusmodi series erui posse, quam passim dedi sive in Comment. Petrop.[1]) sive in Actis Berolin.[2])

2. Nunc autem cum forte cogitationes meae in idem argumentum inciderent, alia ac, ni fallor, multo simplicior et commodior series se mihi obtulit, cuius investigationem ita animo institui.

Considero scilicet quadrantem ellipticum ACB (Fig. 1, p. 359), cuius semiaxes sint $CA = a$, $CB = b$, quibus coordinatae parallelae vocentur

1) L. Euleri Commentatio 52 (indicis Enestroemiani); vide p. 8. A. K.

2) L. Euleri Commentatio 154 (indicis Enestroemiani); vide p. 21. A. K.

$CP = x$ et $PM = y$, ita ut ex natura ellipsis habeatur ista aequatio

$$bbx^2 + aay^2 = aa \cdot bb$$

sive

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

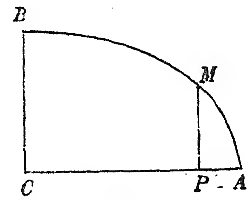

Fig. 1.

ex qua singulari modo definio longitudinem totius arcus AMB sive quartae partis perimetri.

3. Cum igitur esse debeat

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

novam variabilem z in calculum introduco statuendo

$$\frac{x^2}{a^2} = \frac{1+z}{2},$$

ut fiat

$$\frac{y^2}{b^2} = \frac{1-z}{2},$$

unde prodit

$$x = a\sqrt{\frac{1+z}{2}} \quad \text{et} \quad y = b\sqrt{\frac{1-z}{2}}$$

hincque differentiando

$$dx = \frac{adz}{2\sqrt{2(1+z)}} \quad \text{et} \quad dy = \frac{-bdz}{2\sqrt{2(1-z)}};$$

ex quo, si vocemus arcum $BM = s$, statim colligimus

$$ds^2 = dx^2 + dy^2 = \frac{a^2dz^2}{8(1+z)} + \frac{b^2dz^2}{8(1-z)}$$

sive

$$ds^2 = \frac{dz^2}{8}\left(\frac{a^2}{1+z} + \frac{b^2}{1-z}\right) = \frac{dz^2(a^2+b^2-(a^2-b^2)z)}{8(1-z^2)}$$

hincque integrando

$$s = \frac{1}{2\sqrt{2}}\int dz\sqrt{\frac{a^2+b^2-(a^2-b^2)z}{1-z^2}}$$

integrali ita sumto, ut evanescat posito $x = 0$ sive $z = -1$; tum vero integrale extendatur usque ad terminum $x = a$, ubi fit $z = +1$, sicque obtinebitur quaesitus quadrans ellipticus AMB.

4. Quo hanc formulam tractabiliorem reddamus, ponamus brevitatis gratia

$$a^2+b^2=c^2 \quad \text{et} \quad \frac{a^2-b^2}{a^2+b^2}=n.$$

Hoc enim modo consequimur

$$s=\frac{c}{2\sqrt{2}}\int dz\frac{\sqrt{(1-nz)}}{\sqrt{(1-z^2)}},$$

ubi superius radicale more solito in seriem convertamus

$$\sqrt{(1-nz)}=1-\frac{1}{2}nz-\frac{1\cdot 1}{2\cdot 4}n^2z^2-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}n^3z^3-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}n^4z^4-\frac{1\cdot 1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}n^5z^5 \text{ etc.},$$

qui singuli termini nos ad singulares integrationes perducunt; ac bini quidem priores secundum legem datam integrati, ut scilicet evanescant sumto $z=-1$, dabunt

$$\int\frac{dz}{\sqrt{(1-z^2)}}=\text{A. sin. } z-\text{A. sin. }(-1)=\text{A. sin. } z+\frac{1}{2}\pi,$$

$$\int\frac{z\,dz}{\sqrt{(1-z^2)}}=-\sqrt{(1-z^2)}+0;$$

hinc ergo, si sumamus $z=+1$, prodibit

$$\int\frac{dz}{\sqrt{(1-z^2)}}=\pi \quad \text{et} \quad \int\frac{z\,dz}{\sqrt{(1-z^2)}}=0.$$

5. Pro reliquis terminis consideremus reductionem consuetam generalem

$$\int\frac{z^{\lambda+2}dz}{\sqrt{(1-z^2)}}=A\cdot\int\frac{z^{\lambda}dz}{\sqrt{(1-z^2)}}+B\cdot z^{\lambda+1}\sqrt{(1-z^2)},$$

ubi esse oportet

$$A=\frac{\lambda+1}{\lambda+2} \quad \text{et} \quad B=\frac{-1}{\lambda+2},$$

ita ut sit

$$\int\frac{z^{\lambda+2}dz}{\sqrt{(1-z^2)}}=\frac{\lambda+1}{\lambda+2}\int\frac{z^{\lambda}dz}{\sqrt{(1-z^2)}}-\frac{1}{\lambda+2}z^{\lambda+1}\sqrt{(1-z^2)},$$

ubi constantem non adiicimus, quia haec formula iam evanescit sumto

$z = -1$; unde, si iam ponatur $z = +1$, obtinebitur

$$\int \frac{z^{\lambda+2}dz}{\sqrt{(1-z^2)}} = \frac{\lambda+1}{\lambda+2} \int \frac{z^{\lambda}dz}{\sqrt{(1-z^2)}}.$$

6. Ex hac reductione statim liquet omnia integralia ex potestatibus imparibus ipsius z orianda per se evanescere; pro potestatibus autem paribus ad scopum nostrum adipiscimur

$$\int \frac{dz}{\sqrt{(1-z^2)}} = \pi, \quad \int \frac{z^2dz}{\sqrt{(1-z^2)}} = \frac{1}{2}\pi,$$

$$\int \frac{z^4dz}{\sqrt{(1-z^2)}} = \frac{1\cdot 3}{2\cdot 4}\pi, \quad \int \frac{z^6dz}{\sqrt{(1-z^2)}} = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\pi$$

etc.

7. His igitur valoribus substitutis longitudo quadrantis elliptici colligitur fore

$$AMB = \frac{c\pi}{2\sqrt{2}} \left\{ \begin{array}{l} 1 - \frac{1\cdot 1}{2\cdot 4}n^2\cdot\frac{1}{2} - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}n^4\cdot\frac{1\cdot 3}{2\cdot 4} \\ - \frac{1\cdot 1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}n^6\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} \text{ etc.} \end{array} \right\}$$

Pro hac autem forma scribamus tantisper brevitatis gratia

$$AMB = \frac{c\pi}{2\sqrt{2}}\left(1 - \alpha n^2 - \beta n^4 - \gamma n^6 - \delta n^8 - \varepsilon n^{10} \text{ etc.}\right),$$

qui coefficientes sequenti modo succinctius exprimi poterunt

$$\alpha = \frac{1\cdot 1}{2\cdot 4}\cdot\frac{1}{2} = \frac{1\cdot 1}{4\cdot 4}, \quad \frac{\beta}{\alpha} = \frac{3\cdot 5}{8\cdot 8}, \quad \frac{\gamma}{\beta} = \frac{7\cdot 9}{12\cdot 12}, \quad \frac{\delta}{\gamma} = \frac{11\cdot 13}{16\cdot 16} \quad \text{etc.}$$

8. Cum igitur inventi coefficientes tam simplicem et egregiam constituant seriem, haec expressio, quam eruimus, utique maxime videtur attentione digna, cum termini vehementer convergant idque pro omnibus plane ellipsibus, prop-

terea quod semper $\frac{a^2-b^2}{a^2+b^2}=n$ fractio est unitate minor. Habebimus scilicet

$$AMB=\frac{c\pi}{2\sqrt{2}}\left\{\begin{array}{l}1-\frac{1\cdot 1}{4\cdot 4}n^2-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}n^4-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12}n^6\\ \qquad -\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12}\cdot\frac{11\cdot 13}{16\cdot 16}n^8 \text{ etc.}\end{array}\right\}$$

9. Contemplemur hinc casum, quo ellipsis nostra fit circulus radii $=a$; tum enim erit $b=a$, hinc $c=a\sqrt{2}$ et $n=0$, ex quo quadrans circularis prodit, uti quidem notissimum est, $=\frac{1}{2}\pi a$.

10. Deinde vero etiam casus occurrit maxime notatu dignus, quo semiaxis $CB=b=0$; tum enim quadrans ellipticus AMB ipsi semiaxi $CA=a$ fit aequalis; at pro nostra formula erit $c=a$ et $n=1$, quibus valoribus substitutis nanciscimur sequentem aequationem

$$a=\frac{\pi a}{2\sqrt{2}}\left(1-\frac{1\cdot 1}{4\cdot 4}-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\cdot\frac{7\cdot 9}{12\cdot 12}\text{ etc.}\right),$$

qui praecise ipse ille casus est, quo series nostra quam minime est convergens, et qui propterea nostram attentionem eo magis meretur, quod huius seriei summa adcurate assignari potest, cum sit

$$1-\frac{1\cdot 1}{4\cdot 4}-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}\text{ etc. in infin.}=\frac{2\sqrt{2}}{\pi}.$$

10[a][1]). Si cui lubuerit super hac serie calculos numericos instituere, subiungamus hic valores litterarum α, β, γ etc. in fractionibus decimalibus, qui ita se habent

$$\begin{aligned}\alpha&=0{,}0625000\\ \beta&=0{,}0146484\\ \gamma&=0{,}0064087\\ \delta&=0{,}0035798\\ \varepsilon&=0{,}0022821\\ \zeta&=0{,}0015808\\ &\text{etc.,}\end{aligned}$$

1) In editione principe falso numerus 10 iteratur. A. K.

serie autem hucusque tantum continuata prodit

$$1-\alpha-\beta-\gamma-\delta-\varepsilon-\zeta=0{,}9090002;$$

iam vero reperitur $\frac{2\sqrt{2}}{\pi}=0{,}9003200$; unde videmus sequentium litterarum η, ϑ, ι, $\varkappa$ etc. omnium summam efficere debere 0,0086802.

11. Ceterum pro calculo numerico non parum notasse iuvabit nostros coefficientes etiam sequenti modo concinnius exprimi posse

$$\alpha=\frac{1}{16}$$

$$\beta=\frac{1}{64}\cdot\frac{15}{16}$$

$$\gamma=\frac{1}{144}\cdot\frac{15}{16}\cdot\frac{63}{64}$$

$$\delta=\frac{1}{256}\cdot\frac{15}{16}\cdot\frac{63}{64}\cdot\frac{143}{144}$$

$$\varepsilon=\frac{1}{400}\cdot\frac{15}{16}\cdot\frac{63}{64}\cdot\frac{143}{144}\cdot\frac{255}{256}$$

etc.

12. Occasione huius seriei, quam invenimus, operae pretium erit in eius summam a posteriore inquirere, id quod duplici modo fieri potest; prior modus, quem iam olim[1]) proposui ac deinceps saepissime ad usum accommodavi, nos deducit ad aequationem differentialem, cuius integrale per ipsam seriem propositam exprimatur. Quo nunc haec methodus facilius adhiberi queat, ponamus $n=2v$, ut series summanda fiat

$$s=1-\frac{1\cdot 1}{2\cdot 2}v^2-\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}v^4-\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6\cdot 6}v^6 \text{ etc.},$$

1) L. EULERI Commentatio 19 (indicis ENESTROEMIANI): *De progressionibus transcendentibus, seu quarum termini generales algebraice dari nequeunt*, Comment. acad. sc. Petrop. 5 (1730/1), 1738, p. 36; *LEONHARDI EULERI Opera omnia*, series I, vol. 14. A. K.

13. Differentiemus hanc seriem, tum vero iterum per v multiplicemus, ut prodeat

$$\frac{vds}{dv} = -\frac{1\cdot 1}{2}v^2 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4}v^4 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6}v^6 \text{ etc.},$$

quae denuo differentiata praebet

$$\frac{d.vds}{dv^2} = -1\cdot 1v - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^3 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^5 \text{ etc.};$$

hoc scilicet modo ex singulis denominatoribus duos factores sustulimus.

14. Nunc vero denuo ope differentiationis numeratores binis novis factoribus augeamus; hunc in finem primam aequationem in $\sqrt{v}$ ductam differentiemus prodibitque

$$\frac{2d.s\sqrt{v}}{dv} = +v^{-\frac{1}{2}} - \frac{1\cdot 1}{2\cdot 2}5v^{\frac{3}{2}} - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}9v^{\frac{7}{2}} - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6\cdot 6}13v^{\frac{11}{2}} \text{ etc.};$$

haec denuo differentietur et per 2 iterum multiplicando fit

$$\frac{4dd.s\sqrt{v}}{dv^2} = -v^{-\frac{3}{2}} - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^{\frac{1}{2}} - \frac{1\cdot 1}{2.2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^{\frac{5}{2}} \text{ etc.},$$

quae per $v^{\frac{5}{2}}$ multiplicata producit

$$\frac{4v^{\frac{5}{2}}dd.s\sqrt{v}}{dv^2} = -v - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^3 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^5 \text{ etc.};$$

supra vero iam invenimus

$$\frac{d.vds}{dv^2} = -v - \frac{1\cdot 1}{2\cdot 2}3\cdot 5v^3 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}7\cdot 9v^5 \text{ etc.},$$

quae series cum sint aequales, inde deducimus hanc aequationem

$$4v^{\frac{5}{2}}dd.s\sqrt{v} = d.vds,$$

atio continet relationem summae quaesitae s ad variabilem v.

15. Haec ergo aequatio evoluta fit differentiale secundi gradus; sumto enim elemento dv constante ob

$$d.\,s\sqrt{v} = ds\sqrt{v} + \frac{sdv}{2\sqrt{v}}$$

erit

$$dd.\,s\sqrt{v} = dds\cdot\sqrt{v} + \frac{dvds}{\sqrt{v}} - \frac{sdv^2}{4v\sqrt{v}},$$

ergo

$$4v^{\frac{5}{2}}\,dd.\,s\sqrt{v} = 4v^3dds + 4v^2dvds - svdv^2;$$

tum vero ob $d.\,vds = vdds + dvds$ habebitur haec aequatio

$$vdds(1-4v^2) + dvds(1-4v^2) + svdv^2 = 0$$

sive

$$vdds + dvds + \frac{svdv^2}{1-4v^2} = 0.$$

16. Huius igitur aequationis differentialis secundi gradus constructio in nostra est potestate; fiat enim ellipsis, cuius semiaxes sint a et b eiusque peripheriae quarta pars $= q = AMB$; tum vero capiatur

$$c = \sqrt{(a^2+b^2)} \quad \text{et} \quad \frac{a^2-b^2}{a^2+b^2} = n = 2v;$$

unde, cum sit

$$q = \frac{\pi c}{2\sqrt{2}}s,$$

fiet

$$s = \frac{2q\sqrt{2}}{\pi c}.$$

Iam ob $a^2+b^2=c^2$ et $a^2-b^2=2c^2v$ erit

$$a^2 = \frac{c^2(1+2v)}{2} \quad \text{et} \quad b^2 = \frac{c^2(1-2v)}{2}.$$

Quocirca nostra constructio ita erit comparata: sumtis ellipsis semiaxibus

$$a = c\sqrt{\frac{1+2v}{2}} \quad \text{et} \quad b = c\sqrt{\frac{1-2v}{2}}$$

sit q quarta pars perimetri huius ellipsis eritque pro resolutione nostrae aequationis $s = \frac{2q\sqrt{2}}{\pi c}$.

Haec aequatio, si ponamus $s = \frac{z}{\sqrt{v}}$, induet hanc formam simpliciorem

$$ddz + \frac{z\,dv^2}{4v^2(1-4v^2)} = 0,$$

pro qua erit

$$z = \frac{2q\sqrt{2v}}{\pi c}.$$

17. Haec porro aequatio ad differentialem primi gradus reducetur ponendo $z = e^{\int t\,dv}$; tum enim resultabit

$$dt + t^2 dv + \frac{dv}{4v^2(1-4v^2)} = 0,$$

unde si liceret t per v definire, ita ut innotesceret integrale $\int t\,dv$, foret $z = e^{\int t\,dv}$.

18. Hic erat primus modus ex proposita serie infinita in eius summam inquirendi, ubi scilicet loco numeri constantis n quantitatem variabilem v introduximus; altero autem modo idem praestandi, cuius plurima specimina iam passim occurrunt, quantitas constans n talis relinquitur; ponamus autem $n = 2m$, ita ut nostra series summanda sit

$$1 - \frac{1\cdot 1}{2\cdot 2}m^2 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}m^4 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6\cdot 6}m^6 \text{ etc.}$$

19. Nunc fingamus esse

$$s = \int dz \sqrt[4]{(1-2m^2p)},$$

postquam scilicet absoluta integratione quantitati variabili z certus valor determinatus fuerit tributus; litteram vero p etiam ut variabilem spectemus, quae cuiusmodi functio ipsius z capi debeat, ut haec integratio ad nostram seriem infinitam perducatur, sequenti modo investigabimus.

20. Evoluta autem formula irrationali $(1-2m^2p)^{\frac{1}{4}}$ in hanc seriem infinitam

$$1 - \frac{1}{2}m^2p - \frac{1\cdot 3}{2\cdot 4}m^4p^2 - \frac{1\cdot 3\cdot 7}{2\cdot 4\cdot 6}m^6p^3 \text{ etc.}$$

quantitas s sequenti serie formularum integralium definietur

$$s = z - \frac{1}{2}m^2\int p\,dz - \frac{1\cdot 3}{2\cdot 4}m^4\int p^2dz - \frac{1\cdot 3\cdot 7}{2\cdot 4\cdot 6}m^6\int p^3dz \text{ etc.}$$

Nunc vero statuamus, si post singulas integrationes variabili z certus valor determinatus tribuatur, tum fore

$$\int p\,dz = \frac{1}{2}z, \quad \int p^2dz = \frac{5}{4}\int p\,dz,$$

$$\int p^3dz = \frac{9}{6}\int p^2dz, \quad \int p^4dz = \frac{13}{8}\int p^3dz$$

etc.;

sic enim fiet

$$s = z\left(1 - \frac{1\cdot 1}{2\cdot 2}m^2 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}m^4 - \frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{6\cdot 6}m^6 \text{ etc.}\right),$$

quae est ipsa nostra series proposita.

21. Nunc igitur tota quaestio huc redit, cuiusmodi functionem ipsius z pro p sumi oporteat, ut stabilita illa ratio integralium, dum scilicet variabili z certus valor tribuitur, obtineatur; ista autem relatio generatim ita exprimitur

$$\int p^\lambda dz = \frac{4\lambda - 3}{2\lambda}\int p^{\lambda-1}dz;$$

ponamus igitur integralibus adhuc indefinite sumtis fore

$$\int p^\lambda dz = \frac{4\lambda-3}{2\lambda}\int p^{\lambda-1}dz + \frac{p^\lambda\cdot Q}{2\lambda};$$

facta ergo differentiatione prodibit

$$p^\lambda dz = \frac{4\lambda-3}{2\lambda}p^{\lambda-1}dz + \frac{1}{2}p^{\lambda-1}Q\,dp + \frac{p^\lambda}{2\lambda}dQ,$$

quae per $p^{\lambda-1}$ divisa et per 2λ multiplicata praebet

$$2\lambda p\,dz = (4\lambda - 3)dz + \lambda Q\,dp + p\,dQ,$$

et cum haec aequatio subsistere debeat, quicquid sit λ, suppeditat nobis has

duas aequationes

$$2pdz - 4dz - Qdp = 0, \qquad -3dz + pdQ = 0,$$

ex quibus utramque functionem p et Q definire licebit.

22. Perinde autem hic est, sive p et Q sint functiones ipsius z sive z et Q ipsius p, dummodo earum relatio inter se stabiliatur; ex posteriore autem statim habemus

$$dz = \tfrac{1}{3} pdQ,$$

qui valor in priore substitutus praebet

$$\tfrac{2}{3}(p-2)pdQ - Qdp = 0,$$

ex qua fit

$$\frac{dQ}{Q} = \frac{3dp}{2p(p-2)} = -\frac{3dp}{4p} + \frac{3dp}{4(p-2)},$$

unde integrando oritur

$$\log. Q = -\frac{3}{4}\log. p + \frac{3}{4}\log. (p-2) = +\frac{3}{4}\log. \frac{p-2}{p},$$

unde fit

$$Q = 2\left(\frac{p-2}{p}\right)^{\frac{3}{4}};$$

tum vero, quia ex prima aequatione est $dz = \frac{Qdp}{2(p-2)}$, hinc fit

$$dz = \frac{dp}{p^{\frac{3}{4}}(p-2)^{\frac{1}{4}}} = \frac{dp}{\sqrt[4]{p^3(p-2)}}.$$

Nunc autem inprimis observari oportet, ut pro utroque integrationis termino formula algebraica ibi adiecta

$$p^{\lambda} Q = 2p^{\lambda - \frac{3}{4}}(p-2)^{\frac{3}{4}}$$

evanescat, sicque manifestum est integrationis terminos statui debere $p = 0$ et $p = 2$.

23. Ecce ergo formulam nostram integralem initio introductam hoc modo repraesentatam

$$s = \int \frac{dp\sqrt[4]{(1-2m^2p)}}{\sqrt[4]{p^3(p-2)}};$$

quare cum sit

$$z=\int\frac{dp}{\sqrt[4]{p^3(p-2)}},$$

ipsa nostra series proposita

$$1-\frac{1\cdot 1}{2\cdot 2}m^2-\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}m^4 \text{ etc.}$$

aequabitur fractioni $\frac{s}{z}$, postquam scilicet haec integralia ita fuerint sumta, ut evanescant posito $p=0$, tum vero statuatur $p=2$; quamobrem illas duas formulas integrales ita exprimi conveniet

$$s=\int\frac{dp\sqrt[4]{(1-2m^2p)}}{\sqrt[4]{p^3(2-p)}} \quad \text{et} \quad z=\int\frac{dp}{\sqrt[4]{p^3(2-p)}}.$$

24. Ex his igitur series nostra supra inventa

$$1-\frac{1\cdot 1}{4\cdot 4}n^2-\frac{1\cdot 1}{4\cdot 4}\cdot\frac{3\cdot 5}{8\cdot 8}n^4 \text{ etc.},$$

cuius summam iam vidimus esse $\frac{2q\sqrt{2}}{\pi c}$, etiam hoc modo per duas formulas integrales repraesentari potest, quae facta levi mutatione $p=2r$ erunt, ea, quae numeratorem constituit,

$$s=\int\frac{dr\sqrt[4]{(1-nnr)}}{\sqrt[4]{r^3(1-r)}},$$

altera vero, quae constituit denominatorem,

$$z=\int\frac{dr}{\sqrt[4]{r^3(1-r)}};$$

ipsa autem fractio nostram seriem exhibebit; nunc autem termini integrationis $r=0$ et $r=1$.

25. Adhuc succinctius hae formulae transformari possunt sumendo $r=t^4$; tum enim ambae formulae integrales erunt

$$s=\int\frac{dt\sqrt[4]{(1-n^2t^4)}}{\sqrt[4]{(1-t^4)}} \quad \text{et} \quad z=\int\frac{dt}{\sqrt[4]{(1-t^4)}}$$

terminis integrationis existentibus etiamnunc $t=0$ et $t=1$; quo observato fractio $\frac{s}{z}$ aequabitur nostrae seriei sive erit

$$\frac{s}{z}=\frac{2q\sqrt{2}}{\pi c},$$

ubi q denotat quartam partem peripheriae ellipsis, cuius semiaxes sunt

$$c\sqrt{\frac{1+n}{2}} \quad \text{et} \quad c\sqrt{\frac{1-n}{2}}.$$

26. Hinc casu $n=0$ manifesto fit $\frac{s}{z}=1$, casu vero $n=1$ ob $s=t=1$ fiet

$$\frac{1}{z}=\frac{2\sqrt{2}}{\pi} \quad \text{sive} \quad z=\int\frac{dt}{\sqrt[4]{(1-t^4)}}=\frac{\pi}{2\sqrt{2}},$$

quod quidem iam aliunde constat.

SUMMARIUM

Commentationis 28 indicis ENESTROEMIANI

SPECIMEN DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIALIUM SINE INDETERMINATARUM SEPARATIONE[1])

Ex manuscriptis academiae scientiarum Petropolitanae nunc primum editum[2])

Quotiescumque in resolvendo problemate ad aequationem differentialem perventum est, necesse est ad plenariam eius solutionem, ut ista aequatio integretur aut saltem geometrice construatur. At neque integratio neque constructio geometrica facile succedunt, nisi quando aequatio eo sit perducta, ut litterae variabiles seu indeterminatae in quolibet termino aequationis ab invicem seiunctae sint. Hanc ob causam separatio indeterminatarum res maximi momenti est in rebus analyticis. Extant quidem passim methodi particulares integrandi aut construendi aequationes differentiales absque indeterminatarum separatione. Observavit autem Cl. EULERUS iis solum casibus eas methodos succedere, ubi indeterminatarum separatio aut facilis sit aut ex ipsa constructione elici possit. Ut igitur hanc rem magis perficeret, exemplum adducit aequationis, in qua indeterminatae nullo modo separari possunt, atque huiusmodi aequationis constructionem tradit geometricam ope rectificationis ellipsis.

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1) Vide p. 1. A. K.
2) Vide p. X praefationis. A. K.